Theorems

Wednesday, May 9, 2018 12:53 AM

Theorem 1.11: Greatest-Lower-Bound Property

- Suppose S is an ordered set with the least-upper-bound property
- Suppose $B \subset S, B \neq \emptyset$ and B is bounded below
- Let L be the set of lower bounds of B
- Then $\alpha = \sup L$ exists in S and $\alpha = \inf B$

Theorem 1.20: The Archimedean property of $\mathbb{R}$

- Given $x, y \in \mathbb{R}$, and $x > 0$
- There is a positive integer n such that $nx > y$

Theorem 1.20: $\mathbb{Q}$ is dense in $\mathbb{R}$

- If $x, y \in \mathbb{R}$, and $x < y$, then there exists a $p \in \mathbb{Q}$ s.t. $x < p < y$
- We can always find a rational number between two real numbers

Theorem 1.21: n-th Root of Real Numbers

- For every real $x > 0$, and positive integer n
- There is one and only one positive real number y s.t. $y^n = x$
- In this case, we write $y = x^{\frac{1}{n}}$

Theorem 1.31: Properties of Complex Numbers

- If z and w are complex numbers, then
- $\overline{z+w} = \bar{z} + \bar{w}$
- $\overline{zw} = \bar{z} \cdot \bar{w}$
- $z + \bar{z} = 2\text{Re}(z), z - \bar{z} = 2i\ \text{Im}(z)$
- $z\bar{z}$ is real and positive (except when $z = 0$)

Theorem 1.33: Properties of Complex Numbers

- If z and w are complex numbers, then
- $|z| > 0$ unless $z = 0$ in which case $|z| = 0$
- $|\bar{z}| = |z|$
- $|zw| = |z||w|$
- $|\text{Re}(z)| \leq |z|$
- $|z + w| \leq |z| + |w|$ (Triangle Inequality)

Theorem 1.37: Properties of Euclidean Spaces

- Suppose $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n, \alpha \in \mathbb{R}$, then
- $|\vec{x}| \geq 0$

- $|\vec{x}| = 0$ if and only if $\vec{x} = \vec{0}$
- $|\alpha\vec{x}| = |\alpha| \cdot |\vec{x}|$
- $|\vec{x} \cdot \vec{y}| \leq |\vec{x}| \cdot |\vec{y}|$ (Schwarz's Inequality)
- $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$ (Triangle Inequality)
- $|\vec{x} - \vec{y}| \leq |\vec{x} - \vec{z}| + |\vec{y} - \vec{z}|$ (Triangle Inequality)

Theorem 2.8: Infinite Subset of Countable Set

- Every infinite subset of a countable set is countable

Theorem 2.12: Union of Countable Sets

- Let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of countable sets, then
- $S = \bigcup_{n=1}^{\infty} E_n$ is also countable

Theorem 2.13: Cartesian Product of Countable Sets

- Let A be a countable set
- Let B_n be the set of all n-tuples $(a_1, a_2, \dots a_n)$ where
  - $a_k \in A$ for $k = 1,2,\dots,n$
  - a_k may not be distinct
- Then B_n is countable

Theorem 2.14: Cantor's Diagonalization Argument

- Let A be the set of all sequqnecse whose digits are 0 and 1
- Then A is uncountable

Theorem 2.19: Every Neighborhood is an Open Set

- Every neighborhood is an open set

Theorem 2.20: Property of Limit Point

- If p is a limit point of E
- Then every neighborhood of p contains infinitely many points of E

Theorem 2.22: De Morgan's Law

- Let $\{E_\alpha\}$ be a finite or infinite collection of sets, then
- $\left(\bigcup_{\alpha} E_\alpha\right)^c = \bigcap_{\alpha} (E_\alpha)^c$

Theorem 2.23: Complement of Open/Closed Set

- A set E is open if and only if E^c is closed
- Note: This does not say that open is not closed and closed is not open

Theorem 2.24: Intersection and Union of Open/Closed Sets

- For any collection $\{G_n\}$ of open sets, $\bigcup_{\alpha} G_\alpha$ is open
- For any collection $\{F_n\}$ of closed sets, $\bigcap_{\alpha} F_\alpha$ is closed
- For any finite collection, $G_1, G_2, \dots, G_n$ of open sets, $\bigcap_{i=1}^{n} G_i$ is also open
- For any finite collection, $F_1, F_2, \dots, F_n$ of closed sets, $\bigcup_{i=1}^{n} F_i$ is also closed

Theorem 2.27: Properties of Closure

- If X is a metric space and $E \subset X$, then
- $\bar{E}$ is closed
- $E = \bar{E} \Leftrightarrow E$ is closed
- $\bar{E} \subset F$ for every closed set $F \subset X$ s.t. $E \subset F$

Theorem 2.28: Closure and Least Upper Bound Property of $\mathbb{R}$

- If $E \neq \emptyset, E \subset \mathbb{R}$, and E is bouned above, then $\sup E \in \bar{E}$
- Hence $\sup E \in E$ if E is closed

Theorem 2.34: Compact Sets are Closed

- Compact subsets of metric spaces are closed

Theorem 2.35: Closed Subsets of Compact Sets are Compact

- Closed subsets of compact sets are compact

Theorem 2.36: Cantor's Intersection Theorem

- If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X s.t.
- The intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty
- Then $\bigcap_{\alpha} K_\alpha$ is nonempty

Theorem 2.37: Infinite Subset of Compact Set

- If E is an infinite subset of a compact set K
- Then E has a limit point in K

Theorem 2.38: Nested Intervals Theorem

- If $\{I_n\}$ is a sequence of closed intervals in $\mathbb{R}$ s.t. $I_n \supset I_{n+1}, \forall n \in \mathbb{N}$
- Then $\bigcap_{n=1}^{\infty} I_n$ is nonempty

Theorem 2.39: Nested k-cell

- Let k be a positive integer
- If $\{I_n\}$ is a sequence of k-cells s.t. $I_n \supset I_{n+1}, \forall n \in \mathbb{N}$
- Then $\bigcap_{n=1}^{\infty} I_n$ is nonempty

Theorem 2.40: Compactness of k-cell

- Every k-cell is compact

Theorem 2.41: The Heine-Borel Theorem

- For a set $E \subset \mathbb{R}^k$, the following properties are equivalent
- E is closed and bounded
- E is compact
- Every infinite subset of E has a limit point in E

Theorem 2.42: The Weierstrass Theorem

- Every bounded infinite subset E of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$

Theorem 2.47: Connected Subset of $\mathbb{R}$

- $E \subset \mathbb{R}$ is connected if and only if E has the following property
- If $x, y \in E$ and $x < z < y$, then $z \in E$

Theorem 3.2: Important Properties of Convergent Sequences

- Let $\{p_n\}$ be a sequence in a metric space X
- $p_n \to p \in X \Leftrightarrow$ any neighborhood of p contains p_n for all but finitely many n
- Given $p \in X$ and $p' \in X$. If $\{p_n\}$ converges to p and to p', then $p = p'$
- If $\{p_n\}$ converges, then $\{p_n\}$ is bounded
- If $E \subset X$, and $p \in E'$, then there exists a sequence $\{p_n\}$ in E s.t. $p_n \to p$

Theorem 3.3: Algebraic Limit Theorem

- Suppose $\{s_n\}, \{t_n\}$ are complex sequence, and $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$, then
- $\lim_{n\to\infty} s_n + t_n = s + t$
- $\lim_{n\to\infty} c + s_n = c + s, \forall c \in \mathbb{C}$
- $\lim_{n\to\infty} cs_n = cs, \forall c \in \mathbb{C}$
- $\lim_{n\to\infty} s_n t_n = st$
- $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s} \left(s_n \neq 0, \forall n \in \mathbb{N}, \text{and } s \neq 0\right)$

Theorem 3.4: Convergence of Sequence in $\mathbb{R}^n$

- Suppose $\overrightarrow{x_n} = \left(\alpha_{1,n}, \alpha_{2,n}, \dots, \alpha_{k,n}\right) \in \mathbb{R}^k$ where $n \in \mathbb{N}$, then

- $\{\overrightarrow{x_n}\}$ converges to $(\alpha_1, \alpha_2, \dots, \alpha_k) \Leftrightarrow \lim_{n\to\infty} \alpha_{j,n} = \alpha_j \ (1 \le j \le k)$

Theorem 3.6: Properties of Subsequence

- If $\{p_n\}$ is a sequence in a compact metric space X
- Then some subsequence of $\{p_n\}$ converges to a point of X
- Every bounded sequences in $\mathbb{R}^k$ contains a convergent subsequence

Theorem 3.10: Diameter and Closure

- If $\bar{E}$ is the closure of a set E in a metric space X, then diam $\bar{E} =$ diam E

Theorem 3.10: Nested Compact Set

- If K_n is a sequence of compact sets in X s.t.
- $K_n \supset K_{n+1}, \forall n$ and $\lim_{n\to\infty}$ diam $K_n = 0$
- Then $\bigcap_{n=1}^{\infty} K_n$ consists of exactly one point

Theorem 3.11: Cauchy Sequence and Convergence

- In any metric space X, every convergent sequence is a Cauchy sequence
- If X is a compact metric space and $\{p_n\}$ is a Cauchy sequence
- Then $\{p_n\}$ converges to some point of X
- In $\mathbb{R}^k$, every Cauchy sequence converges

Theorem 3.14: Monotone Convergence Theorem

- If $\{s_n\}$ is monotonic, then $\{s_n\}$ converges if and only if it is bounded

Theorem 3.17: Properties of Upper Limits

- Let $\{s_n\}$ be a sequence of real numbers, then
- $s^* \in E$
- If $x > s^*$, then $\exists N \in \mathbb{N}$ s. t. $s_n < x$ for $n \ge N$
- Moreover s^* is the only number with these properties

Theorem 3.20: Some Special Sequences

- If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$
- If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$
- $\lim_{n\to\infty} \sqrt[n]{n} = 1$
- If $p > 0, \alpha \in \mathbb{R}$, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$
- If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$

Theorem 3.22: Cauchy Criterion for Series

- $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\left|\sum_{k=n}^{m} a_k\right| < \varepsilon, \forall m \geq n \geq N$

Theorem 3.23: Series and Limit of Sequence

- If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

Theorem 3.24: Convergence of Monotone Series

- A series of nonnegative real numbers converges if and only if
- its partial sum form a bounded sequence

Theorem 3.25: Comparison Test

- If $|a_n| < c_n$ for $n \geq N_0 \in \mathbb{N}$ and $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges
- If $a_n \geq d_n \geq 0$ for $n \geq N_0 \in \mathbb{N}$ and $\sum_{n=1}^{\infty} d_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges

Theorem 3.26: Convergence of Geometric Series

- If $0 < x < 1$, then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$
- If $x > 1$, the series diverges

Theorem 3.27: Cauchy Condensation Test

- Suppose $a_1 \geq a_2 \geq \cdots \geq 0$, then
- $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + \cdots$ converges

Theorem 3.28: Convergence of p −Series

- $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$

Theorem 3.33: Root Test

- Given $\sum_{n=1}^{\infty} a_n$, put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$, then
- If $\alpha < 1$, $\sum_{n=1}^{\infty} a_n$ converges
- If $\alpha > 1$, $\sum_{n=1}^{\infty} a_n$ diverges
- If $\alpha = 1$, this test gives no information

Theorem 3.34: Ratio Test

- $\sum_{n=1}^{\infty} a_n$ converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$
- $\sum_{n=1}^{\infty} a_n$ diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1, \forall n \geq n_0$ for some fixed $n_0 \in \mathbb{N}$

Theorem 3.39: Convergence of Power Series

- Given the power sires $\sum_{n=1}^{\infty} c_n z^n$
- Put $\alpha \coloneqq \limsup_{n\to\infty} \sqrt[n]{|c_n|}$
- Let $R \coloneqq \frac{1}{\alpha}$ (If $\alpha = +\infty, R = 0;$ If $\alpha = 0, R = +\infty$)
- Then $\sum_{n=1}^{\infty} c_n z^n$ converges if $|z| < R$ and diverges if $|z| > R$

Theorem 3.43: Alternating Series Test

- Suppose we have a real sequence $\{c_n\}$ s.t.
  - $|c_1| \geq |c_2| \geq |c_3| \geq \cdots$
  - $c_{2m-1} \geq 0, c_{2m} \leq 0, \forall m \in \mathbb{N}$
  - $\lim_{n\to\infty} c_n = 0$
- Then $\sum_{n=1}^{\infty} c_n$ converges

Theorem 3.45: Property of Absolute Convergence

- If Σa_n converges absolutely, then Σa_n converges

Theorem 3.54: Riemann Series Theorem

- Let Σa_n be a series of real number which converges nonabsolutely
- Let $-\infty \leq \alpha \leq \beta \leq +\infty$
- Then there exists a rearrangement $\Sigma a_n'$ s.t.
- $\liminf_{n\to\infty} s_n' = \alpha, \limsup_{n\to\infty} s_n' = \beta$

Theorem 3.55: Rearrangement and Absolute Convergence

- If Σa_n is a series of complex numbers which converges absolutely
- Then every rearrangement of Σa_n converges to the same sum

Theorem 4.4: Algebraic Limit Theorem of Functions

- Let X be a metric space, and $E \subset X$
- Suppose p be a limit point of E

- Let f, g be complex functions on E where
  - $\lim_{x \to p} f(x) = A$ and $\lim_{x \to p} g(x) = B$
- Then
  - $\lim_{x \to p}(f + g)(x) = A + B$
  - $\lim_{x \to p}(f - g)(x) = A - B$
  - $\lim_{x \to p}(fg)(x) = AB$
  - $\lim_{x \to p}\left(\frac{f}{g}\right)(x) = \frac{A}{B}$ where $B \neq 0$

Theorem 4.6: Continuity and Limits

- In the context of Definition 4.5, if p is also a limit point of E, then
- f is continious at p if and only if $\lim_{x \to p} f(x) = f(p)$

Theorem 4.7: Composition of Continuous Function

- Suppose X, Y, Z are metric spaces, $E \subset X$, $f: E \to Y$, $g: f(E) \to Z$, and
- $h: E \to Z$ defined by $h(x) = g(f(x)), \forall x \in E$
- If f is continuous at $p \in E$, and g is continuous at $f(p)$
- Then h is continuous at p

Theorem 4.8: Characterization of Continuity

- Given metric spaces X, Y
- $f: X \to Y$ is continuous if and only if
- $f^{-1}(V)$ is open in X for every open set $V \subset Y$

Theorem 4.14: Continuous Functions Preserve Compactness

- Statement
  - Let X, Y be metric spaces, X compact
  - If $f: X \to Y$ is continuous, then $f(X)$ is also compact

Theorem 4.15: Applying Theorem 4.14 to $\mathbb{R}^k$

- Let X be a compact metric space
- If $f: X \to \mathbb{R}^k$ is continuous, then $f(X)$ is closed and bounded
- Thus, f is bounded

Theorem 4.16: Extreme Value Theorem

- Let f be a continuous real function on a compact metric space X
- Let $M \coloneqq \sup_{p \in X} f(p)$, and $m \coloneqq \inf_{p \in X} f(p)$
- Then $\exists p, q \in X$ s.t. $f(p) = M$ and $f(q) = m$
- Equivalently, $\exists p, q \in X$ s.t. $f(q) \leq f(x) \leq f(p), \forall x \in X$

Theorem 4.17: Inverse of Continuous Bijection is Continuous

- Let X, Y be metric spaces, X compact
- Suppose $f: X \to Y$ is continuous and bijective
- Define $f^{-1}: Y \to X$ by $f^{-1}(f(x)) = x, \forall x \in X$
- Then f^{-1} is also continuous and bijective

Theorem 4.19: Uniform Continuity and Compactness

- Let X, Y be metric spaces, X compact
- If $f: X \to Y$ is continuous, then f is also uniformly continuous

Theorem 4.20: Continuous Mapping from Noncompact Set

- Let E be noncompact set in $\mathbb{R}$
- Then there exists a continuous function f on E s.t.
  - f is not bounded
  - f is bounded but has no maximum
  - E is bounded, but f is not uniformly continuous

Theorem 4.22: Continuous Mapping of Connected Set

- Let X, Y be metric spaces
- Let $f: X \to Y$ be a continuous mapping
- If $E \subset X$ is connected then $f(E) \subset Y$ is also connected

Theorem 4.23: Intermediate Value Theorem

- Let $f: \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$
- If $f(a) < f(b)$ and if c statifies $f(a) < c < f(b)$
- Then $\exists x \in (a, b)$ s.t. $f(x) = c$

Theorem 5.2: Differentiability Implies Continuity

- Let f be defined on $[a, b]$
- If f is differentiable at $x \in [a, b]$ then f is continuous at x

Theorem 5.5: Chain Rule

- Given
  - f is continuous on $[a, b]$, and $f'(x)$ exists at $x \in [a, b]$
  - g is defined on $I \supset \operatorname{im}(f)$, and g is differentiable at $f(x)$
- If $h(t) = g(f(t))$ $(a \le t \le b)$, then
  - h is differentiable at x, and $h'(x) = g'(f(x)) \cdot f'(x)$

Theorem 5.8: Local Extrema and Derivative

- Let f be defined on $[a, b]$
- If f has a local maximum (or minimum) at $x \in (a, b)$

- Then $f'(x) = 0$ if it exists

Theorem 5.9: Extended Mean Value Theorem

- Given
  - f and g are continuous real-valued functions on $[a, b]$
  - f, g are differentiable on (a, b)
- Then there is a point $x \in (a, b)$ at which
  - $[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$

Theorem 5.10: Mean Value Theorem

- Let $f: [a, b] \to \mathbb{R}$
- If f is continuous on $[a, b]$ and differentiable on (a, b)
- Then $\exists x \in (a, b)$ s.t. $f(b) - f(a) = (b - a)f'(x)$

Theorem 5.11: Derivative and Monotonicity

- Suppose f is differentiable on (a, b)
- If $f'(x) \geq 0, \forall x \in (a, b)$, then f is monotonically increasing
- If $f'(x) = 0, \forall x \in (a, b)$, then f is constant
- If $f'(x) \leq 0, \forall x \in (a, b)$, then f is monotonically decreasing

Theorem 5.15: Taylor's Theorem

- Suppose
  - f is a real-valued function on $[a, b]$
  - Fix a positive integer n
  - $f^{(n-1)}$ is continuous on (a, b)
  - $f^{(n)}(t)$ exists $\forall t \in (a, b)$
- Let $\alpha, \beta \in [a, b]$, where $a \neq \beta$
- Define $P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k$
- Then $\exists x$ between α and β s.t.
- $f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$

Theorem 6.4: Properties of Refinement

- If P^* is a refinement of P, then
- $L(P, f, \alpha) \leq L(P^*, f, \alpha)$
- $U(P^*, f, \alpha) \leq U(P, f, \alpha)$

Theorem 6.5: Properties of Common Refinement

- $\overline{\int_a^b} f dx \leq \underline{\int_a^b} f dx$

Theorem 6.6

- $f \in \mathcal{R}(\alpha)$ on $[a,b]$ if and only if
- $\forall \varepsilon > 0$, there exists a partition P s.t. $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$

Theorem 6.8

- If f is continuous on $[a,b]$, then $f \in \mathcal{R}(\alpha)$ on $[a,b]$

Theorem 6.9

- If f is monotonic on $[a,b]$, and α is continuous on $[a,b]$
- Then $f \in \mathcal{R}(\alpha)$ on $[a,b]$

Theorem 6.10

- If f is bounded on $[a,b]$ with finitely many points of discontiunity
- And α is continuous on these points, then $f \in \mathcal{R}(\alpha)$

Theorem 6.20: Fundamental Theorem of Calculus (Part I)

- Let $f \in \mathcal{R}$ on $[a,b]$
- Define $F(x) = \int_a^x f(t)dt$ for $x \in [a,b]$, then
  - F is continuous on $[a,b]$
- Furthermore, if f is continuous at $x_0 \in [a,b]$, then
  - F is differentiable at x_0, and
  - $F'(x_0) = f(x_0)$

Theorem 6.21: Fundamental Theorem of Calculus (Part II)

- Let $f \in \mathcal{R}$ on $[a,b]$
- If there exists a differentiable function F on $[a,b]$ s.t. $F' = f$
- Then $\int_a^b f(x)dx = F(b) - F(a)$

Number Systems, Irrationality of $\sqrt{2}$

Wednesday, January 24, 2018 12:01 PM

Course Overview

- The real number system
- Metric spaces and basic topology
- Sequences and series
- Continuity
- Topics from differential and integral calculus

Grading

Homework assignments	20%
Quiz (Feb. 9)	5%
Midterm 1 (Mar. 9)	20%
Midterm 2 (Apr. 13)	20%
Final (May 10 @ 7:45-9:45 AM)	35%

A	≥90%
B	≥80%
C	≥70%
D	≥60%

Tutoring

- Tom Stone @VV B205
- Monday 2:30 - 4:30 PM
- Tuesday 2:00 - 4:00 PM

Number Systems

- Natural Numbers: $\mathbb{N} = \{1,2,3,\dots\}$
- Integers: $\mathbb{Z} = \{0,\pm 1,\pm 2,\pm 3,\dots\}$
- Rational Numbers: $\mathbb{Q} = \left\{\frac{a}{b} \middle| a,b \in \mathbb{Z}, b \neq 0\right\}$
- Real numbers $\mathbb{R}$: fill the "holes" in the rational numbers

Example 1.1: Irrationality of $\sqrt{2}$

- There is no rational number p such that $p^2 = 2$
- Proof by contradiction
- Assume there is a rational number p such that $p^2 = 2$
- Then $p = \dfrac{m}{n}$, where $m, n \in \mathbb{Z}, n \neq 0$, and m, n have no common factor

- $\left(\frac{m}{n}\right)^2 = 2 \Rightarrow \frac{m^2}{n^2} = 2 \Rightarrow m^2 = 2n^2$
- So m is even
- $m = 2k\ (k \in \mathbb{Z}) \Rightarrow (2k)^2 = 2n^2 \Rightarrow 4k^2 = 2n^2 \Rightarrow 2k^2 = n^2$
- So n is also even
- m, n are both division by 2
- This contradicts the fact that m, n have no common factor
- So no such p exists

Sets, Gaps in Q, Field

Friday, January 26, 2018 12:03 PM

Definition 1.3: Sets

- Contains
  - If A is a set and x **is an element** of A, then we write $x \in A$
  - Otherwise, we write $x \notin A$
- Set
  - The **empty set** or **null set** is a set with no elements, and is denoted as $\emptyset$
  - If a set has at least one element, it is called **nonempty**
- Subset
  - If A and B are sets and every element of A is an element of B
  - Then A is a **subset** of B
  - Rubin write this $A \subset B$, or $\mathrm{B} \supset \mathrm{A}$
  - Fact: $A \subset A$ for all sets A
- Proper subset
  - If B contain something not in A, then A is a **proper subset** of B
- Equal
  - If $A \subset B$ and $B \subset A$ then $A = B$
  - Otherwise $A \neq B$

Example 1.1: Gaps in Rational Number System

- We have proved that $\sqrt{2}$ is not rational
- i.e. there is no rational number p such that $p^2 = 2$
- Let $A = \{p \in \mathbb{Q} | p^2 < 2\}, B = \{p \in \mathbb{Q} | p^2 > 2\}$
- Prove: **A has no largest element**, and **B has no smallest element**
  - Let $p \in \mathbb{Q}$, and $p > 0$
  - Let $q \coloneqq p - \dfrac{p^2 - 2}{p + 2} = \dfrac{2p + 2}{p + 2}$
  - Then $q^2 - 2 = \left(\dfrac{2p + 2}{p + 2}\right)^2 - 2 = \dfrac{2(p^2 - 2)}{(p + 2)^2}$
  - If $p \in \mathrm{A}$
    - then $p^2 - 2 < 0$
    - $\Rightarrow q^2 - 2 = \dfrac{2(p^2 - 2)}{(p + 2)^2} < 0$
    - $\Rightarrow q^2 < 2$

- $\Rightarrow q \in \mathrm{A}$
- $\Rightarrow q > p$
- i.e. A has no largest element

- If $p \in \mathrm{B}$
  - then $p^2 - 2 > 0$
  - $\Rightarrow q^2 - 2 = \dfrac{2(p^2-2)}{(p+2)^2} > 0$
  - $\Rightarrow q^2 > 2$
  - $\Rightarrow q \in \mathrm{B}$
  - $\Rightarrow q < p$
  - i.e. B has no smallest element

Definition 1.12: Field

- A field is a set $\mathbb{F}$ with **two binary operations** called **addition** and **multiplication**
- that satisfy that following field axioms
  - Axioms for addition $(+)$
    - (A1) If $x \in \mathbb{F}$ and $y \in \mathbb{F}$, then $x + y \in \mathbb{F}$
    - (A2) Addition is communicate: $x + y = y + x, \forall x, y \in \mathbb{F}$
    - (A3) Addition is associative: $(x + y) + z = x + (y + z), \forall x, y, z \in \mathbb{F}$
    - (A4) There exists $0 \in \mathbb{F}$ s.t. $x + 0 = x, \forall x \in \mathbb{F}$
    - (A5) $\forall x \in \mathbb{F}$, there exists an additive inverse $-x \in \mathbb{F}$ s.t. $x + (-x) = 0$
  - Axioms for multiplication $(\times \text{ or } \cdot)$
    - (M1) If $x \in \mathbb{F}$ and $y \in \mathbb{F}$, then $xy \in \mathbb{F}$
    - (M2) Addition is communicate: $xy = yx, \forall x, y \in \mathbb{F}$
    - (M3) Addition is associative: $(xy)z = x(yz), \forall x, y, z \in \mathbb{F}$
    - (M4) $\mathbb{F}$ contains an element $1 \neq 0$ s.t. $1 \cdot x = x, \forall x \in \mathbb{F}$
    - (M5) If $x \in \mathbb{F}$ and $x \neq 0$, then there exists $\frac{1}{x} \in \mathbb{F}$ s.t. $x \cdot \frac{1}{x} = 1$
  - (D) The distributive law: $x(y + z) = xy + xz, \forall x, y, z \in \mathbb{F}$
- Example
  - The real numbers are an example of field

Field, Order, Ordered Set

Monday, January 29, 2018 12:00 PM

Proposition 1.14: Properties of Fields (Addition)

- Given a field $\mathbb{F}$, for $x, y, z \in \mathbb{F}$
  (1) If $x + y = x + z$, then $y = z$
    - $x + y = x + z$
    - $(x + y) + (-x) = (x + z) + (-x)$ by (A5)
    - $x + y + (-x) = x + z + (-x)$ by (A3)
    - $x + (-x) + y = x + (-x) + z$ by (A2)
    - $0 + y = 0 + z$ by (A6)
    - $y = z$ by (A4)

  (2) If $x + y = x$, then $y = 0$
    - $x + y = x = x + 0$
    - $\Rightarrow y = 0$

  (3) If $x + y = 0$, then $y = -x$
    - $x + y = 0 = x + (-x)$
    - $\Rightarrow y = -x$

  (4) $-(-x) = x$
    - $(-x) + (-(-x)) = 0$
    - $x + (-x) + (-(-x)) = x + 0$
    - $0 + (-(-x)) = x + 0$
    - $-(-x) = x$

Proposition 1.15: Properties of Fields (Multiplication)

- Given a field $\mathbb{F}$, for $x, y, z \in \mathbb{F}$
  (1) If $x \neq 0$ and $xy = xz$, then $y = z$
  (2) If $x \neq 0$ and $xy = x$, then $y = 1$
  (3) If $x \neq 0$ and $xy = 1$, then $y = \frac{1}{x}$
  (4) If $x \neq 0$, then $\frac{1}{1/x} = x$
- Proof similar to Proposition 1.14

Proposition 1.16: Properties of Fields

- Given a field $\mathbb{F}$, for $x, y \in \mathbb{F}$
  (1) $0x = 0$
    - $0 + 0 = 0$
    - $(0 + 0)x = 0x$

- $0x + 0x = 0x$
- $0x + 0x + \big(-(0x)\big) = 0x + \big(-(0x)\big)$
- $0x = 0$

(2) If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$

- Suppose $x \neq 0, y \neq 0$, but $xy = 0$
- $x \neq 0$, so $\frac{1}{x}$ exists
- $\frac{1}{x}(xy) = \frac{1}{x} \cdot 0$
- $\left(\frac{1}{x} \cdot x\right) y = \frac{1}{x} \cdot 0$
- $1 \cdot y = 0$
- $y = 0$
- This is a contradiction, so $xy \neq 0$

(3) $(-x)y = -(xy) = x(-y)$

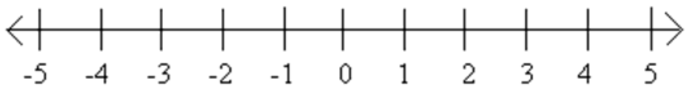
- $(-x)y + xy = \big((-x) + x\big)y = 0 \cdot y = 0$
- $(-x)y + xy + (-xy) = 0 + (-xy)$
- $(-x)y = -xy$
- And the rest is similar

(4) $(-x)(-y) = xy$

- Use (3), $(-x)(-y) = -\big(x(-y)\big) = -(-xy) = xy$

Definition 1.5: Order

- Intuition
  - The real number line



- Definition
  - Let S be a set.
  - An **order** on S is a relation, denoted by $<$
  - with the following two properties:
    - If $x, y \in S$, then only one of the statements $x < y, x = y, y < x$ is true
    - If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$ (Transitivity)
- Other notations
  - $x \leq y$ means either $x < y$ or $x = y$
  - $x \geq y$ means either $x > y$ or $x = y$

Definition 1.6: Ordered Set

- Definition

- An **ordered set** is a set for which an order is defined.

- Example
    - $\mathbb{Q}$ is an ordered set under the definition that
    - for $r, s \in \mathbb{Q}, r < s$ if and only if $s - r$ is positive

Infimum and Supremum, Ordered Field

Wednesday, January 31, 2018 12:00 PM

Definition 1.7: Upper Bound and Lower Bound

- Suppose S is an ordered set and $E \subset S$
- If there exists $\beta \in S$ such that $x \leq \beta, \forall x \in E$
- We say that x is bounded above and call β an **upper bound** for E
- If there exists $\beta \in S$ such that $x \geq \beta, \forall x \in E$
- We say that x is bonded below by β, and β is a **lower bound** for E

Definition 1.8: Least Upper Bound and Greatest Lower Bound

- Definition
  - Suppose S is an ordered set and $E \subset S$ is bounded above.
  - Suppose there exists $\alpha \in S$ s.t.
    - α is an upper bond of E
    - If $\gamma < \alpha$, then γ is not an upper bound of E
  - Then we call α the **least upper bound** (or lub or sup or supremium) of E
  - Suppose there exists $\alpha \in S$ s.t.
    - α is an lower bond of E
    - If $\gamma > \alpha$, then γ is not an lower bound of E
  - Then we call α the **greastst lower bound** (or glb or inf or infimum) of E

Examples 1.9: Least Upper Bound and Greatest Lower Bound

- Recall
  - $A = \{q \in \mathbb{Q} | q^2 < 2\}$ has no sup in $\mathbb{Q}$
  - $B = \{q \in \mathbb{Q} | q^2 > 2\}$ has no inf in $\mathbb{Q}$
- If $\alpha = \sup E$ exists, α may or may not be in E
  - $E_1 := \{r \in \mathbb{Q} | r < 0\}$
    - $\inf E_1$ doesn't exist
    - $\sup E_1 = 0 \notin E_1$
  - $E_2 := \{r \in \mathbb{Q} | r \leq 0\}$
    - $\inf E_2$ doesn't exist
    - $\sup E_2 = 0 \in E_2$
  - $E := \left\{\frac{1}{n} \middle| n \in \mathbb{N}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$
    - $\inf E = 0 \notin E$
    - $\sup E = 1 \in E$

Definition 1.10: Least-Upper-Bound property

- We say that a ordered set S has **least-upper-bound property** provided that
- if $E \in S$ s.t. $E \neq \emptyset$ and E is bounded above, then $\sup E$ exists and $\sup E \in S$

Theorem 1.11: Greatest-Lower-Bound Property

- Statement
  - Suppose S is an ordered set with the **least-upper-bound property**
  - Suppose $\boldsymbol{B \subset S, B \neq \emptyset}$ **and** $\boldsymbol{B}$ **is bounded below**
  - Let L be the set of lower bounds of B
  - Then $\boldsymbol{\alpha = \sup L}$ **exists in** $\boldsymbol{S}$ and $\boldsymbol{\alpha = \inf B}$
- Proof
  - $L \neq \emptyset$
    - B is bounded below, so L is not empty
  - L is bounded above
    - Given $b \in B$ and $l \in L$, we have $l \leq b$ by definition of L
    - Therefore, L is bounded above
  - $\sup L$ exists in S
    - $L \neq \emptyset$, L is bounded above
    - And S has least upper bound property
    - So $\sup L$ exists
    - Let $\alpha = \sup L \in S$
  - α is a lower bound for B (i.e. $\alpha \in L$)
    - If $\gamma < \alpha$, then γ is not an upper bound for L, so $\gamma \notin B$
    - So $\alpha \leq x$ for all $x \in B$
    - Thus, α is a lower bound for B
    - i.e. $\alpha \in L$
  - $\alpha = \inf B$
    - If $\beta > \alpha$ is another lower bound for B
    - Then $\beta \notin L$ since α is an upper bound for L
    - So, $\alpha \in L$, but $\beta \notin L$ if $\beta > \alpha$
    - Therefore α is the least upper bound of B
    - i.e. $\alpha = \inf B$
  - Therefore $\alpha = \sup L = \inf B \in S$

Definition 1.17: Ordered Field

- Definition
  - An **ordered field** is a field $\mathbb{F}$ which is also an ordered set, such that
    - $x + y < x + z$ if $x, y, z \in \mathbb{F}$ and $y < z$

        - $xy > 0$ if $x, y \in \mathbb{F}$, $x > 0$ and $y > 0$
    - If $x > 0$, we call x **positive**
    - If $x < 0$, we call x **negative**
- Examples
    - $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$
- Note
    - $\mathbb{R}$ is an ordered field with least-upper-bound property

Ordered Field, Archimedean Property, $\mathbb{Q}$ is dense in $\mathbb{R}$

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Proposition 1.18: Properties of Ordered Field

- Let $\mathbb{F}$ be an ordered field, for $x, y, z \in \mathbb{F}$
  (1) If $x > 0$ then $-x < 0$, and vice versa
    - $x > 0$
    - $x + (-x) > 0 + (-x)$
    - $0 > -x$■
  (2) If $x > 0$ and $y < z$ then $xy < xz$
    - $x > 0, z - y > 0$
    - $x(z - y) > 0$
    - $xz - xy > 0$
    - $xy < xz$■
  (3) If $x < 0$ and $y < z$ then $xy > xz$
    - $x < 0$
    - By (1), $-x > 0$
    - By (2), $(-x)y < (-x)z$
    - $0 < (-x)(z - y)$
    - By (1), $x(z - y) < 0$
    - $xz < xy$■
  (4) If $x \neq 0$ then $x^2 > 0$. In particular $1 > 0$
    - If $x > 0$, by (2), $x^2 > 0 \cdot x = 0$
    - If $x < 0$, by (3), $x^2 > 0 \cdot x = 0$
    - $1 = 1^2 = 1 \times 1 > 0$
    - So $1 > 0$■
  (5) If $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$
    - If $y > 0$, then $\frac{1}{y} \cdot y = 1 > 0 = 0 \cdot \frac{1}{y}$ by (4)
    - So, $\frac{1}{y}$ must have been positive by (2)
    - Similarly, $\frac{1}{x} > 0$
    - Therefore $\left(\frac{1}{x}\right)\left(\frac{1}{y}\right) > 0$

        - Multiply both sides of $x < y$ by $\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)$
        - We get $\frac{1}{y} < \frac{1}{x}$
        - Therefore $0 < \frac{1}{y} < \frac{1}{x}$ ■

Theorem 1.19: Least-Upper-Bound Property of $\mathbb{R}$

- There exists an ordered filed with the least-upper-bond property called $\mathbb{R}$
- Moreover $\mathbb{R}$ has $\mathbb{Q}$ as a subfield
- Proof: See appendix

Theorem 1.20: The Archimedean property of $\mathbb{R}$

- Statement
    - Given $x, y \in \mathbb{R}$, and $x > 0$
    - **There is a positive integer n such that $nx > y$**
- Proof
    - Let $A = \{nx | n \text{ is a positive integer}\}$
    - Assume the Archimedean property is false
    - Then A has an upper bound
    - i.e. $\sup A$ exists
    - Let $\alpha = \sup A$
    - $x > 0$, so $\alpha - x < \alpha$
    - And $\alpha - x$ is not an upper bound for A
    - By definition of $A = \{nx | n \text{ is a positive integer}\}$
    - $\alpha - x < mx$ for some positive integer m
    - So, $\alpha < mx + x = (m + 1)x \in A$
    - This contradicts $\alpha = \sup A$
    - Therefore the Archimedean property is true
- Corollary
    - Given $x > 0$
    - Let $y = 1$, then
    - $\exists n \in \mathbb{Z}_+$ s.t. $nx > 1$
    - Therefore given $x > 0, \exists n \in \mathbb{Z}_+$ s.t. $\frac{1}{n} < x$

Theorem 1.20: $\mathbb{Q}$ is dense in $\mathbb{R}$

- Statement
    - If $x, y \in \mathbb{R}$, and $x < y$, then there exists a $p \in \mathbb{Q}$ s.t. $x < p < y$
    - We can always find a **rational number between two real numbers**

- Proof
    - Let $x, y \in \mathbb{R}$, and $x < y$
    - So $y - x > 0$
    - By the Archimedean property of $\mathbb{R}$
        - There exists a positive integer n s.t.
        - $n(y - x) > 1$
        - $\Rightarrow ny - nx > 1$
        - $\Rightarrow ny > nx + 1$
    - By the Archimedean property of $\mathbb{R}$ again
        - There are positive integers m_1, m_2 s.t.
        - $m_1 > nx, m_2 > -nx$
        - i.e. $-m_2 < nx < m_1$
        - So there is an integer m s.t.
        - $-m_2 \leq m \leq m_1$
        - And more importantly, $m - 1 \leq nx < m$
    - Combining two parts together, we have
        - $nx < m \leq 1 + nx < ny$
        - In particular, $nx < m < ny$
        - Since $n > 0$, we can multiply by $\frac{1}{n}$ and get
        - $\frac{1}{n}(nx) < \frac{1}{n}(m) < \frac{1}{n}(ny)$
        - Therefore $x < q < y$, where $q = \frac{m}{n} \in \mathbb{Q}$

n-th Root of Real Numbers

Monday, February 5, 2018 12:10 PM

Theorem 1.21: n-th Root of Real Numbers

- Notation
  - For a positive integer n
    - $x^n \coloneqq \underbrace{x \cdot x \cdot x \cdots x}_{n \text{ times}}$
  - For a negative integer n
    - $x^n \coloneqq \underbrace{\left(\frac{1}{x}\right) \cdot \left(\frac{1}{x}\right) \cdot \left(\frac{1}{x}\right) \cdots \left(\frac{1}{x}\right)}_{-n \text{ times}}$
- Statement
  - For every real $x > 0$, and positive integer n
  - There is **one and only one positive real number y s.t. $y^n = x$**
  - In this case, we write $y = x^{\frac{1}{n}}$
- Intuition
  - Try this for $n = 2$ and $x = 2$, so $y = \sqrt{2}$
- Proof (Uniqueness)
  - If there were y_1 and y_2 s.t.
  - $y_1^n = x, y_2^n = x$, but $y_1 \neq y_2$
  - Without loss of generality, assume $y_1 < y_2$
  - Then $y_1^n < y_2^n$, so they can't both equal x
  - So, there is at most one such y
- Lemma
  - If n is a positive integer, then
    - $b^n - a^n = (b - a)\left(b^{n-1} + ab^{n-2} + \cdots + a^{n-2}b + a^{n-1}\right)$
  - Moreover, if $b > a > 0$, then
    - $b^n - a^n < (b - a)\underbrace{\left(b^{n-1} + b^{n-1} + \cdots + b^{n-1} + b^{n-1}\right)}_{n \text{ terms}}$
    - $b^n - a^n < (b - a)nb^{n-1}$
- Proof (Existence)
  - Let $E \coloneqq \{t \in \mathbb{R} | t > 0 \text{ and } t^n < x\}$
  - E is not empty
    - Let $t \coloneqq \frac{x}{x+1}$, then $0 < t < 1$ and $t < x$
    - So, $0 < t^n < t < x$

    - Thus, $t \in E$
    - Therefore E is not empty
  - E is bounded above
    - Let $t \in \mathbb{R}$ s.t. $t > 1 + x$
    - Therefore $t^n > t > 1 + x > x$
    - So $t \notin E$ and E is bounded above by $1 + x$
    - By least upper bound property, $\sup E$ exists
    - Let $y \coloneqq \sup E$
  - We now show that $y^n \nless x$ and $y^n \ngtr x$
  - Assume $y^n < x$
    - Choose $h \in \mathbb{R}$ s.t.
    - $0 < h < 1$ and $h < \dfrac{x - y^n}{n(y+1)^{n-1}}$
    - Then $hn(y+1)^{n-1} < y^n$
    - Use the lemma $b^n - a^n < (b - a)nb^{n-1}$
    - Set $a \coloneqq y, b \coloneqq y + h$
    - $(y+h)^n - y^n < (y + h - y)n(y+h)^{n-1}$
    - $(y+h)^n - y^n < hn(y+1)^{n-1}$
    - $(y+h)^n - y^n < y^n$
    - $(y+h)^n < x$
    - Since $y + h > h$ and $y + h \in E$
    - y is not an upper bound of E
    - This contradicts $y = \sup E$
    - Thus, $y^n \nless x$
  - Assume $y^n > x$
    - Let $k \coloneqq \dfrac{y^n - x}{ny^{n-1}} > 0$
    - $k = \dfrac{y^n - x}{ny^{n-1}} = \dfrac{y^n}{ny^{n-1}} - \dfrac{x}{ny^{n-1}} < \dfrac{y^n}{ny^{n-1}} = \dfrac{y}{n} < y$
    - Thus, $0 < k < y$
    - Let $t \in \mathbb{R}$ s.t. $t \geq y - k$, then
    - $y^n - t^n \leq y^n - (y-k)^n$
    - Use the lemma $b^n - a^n < (b - a)nb^{n-1}$
    - Set $a \coloneqq y, b \coloneqq y - k$, then
    - $y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x$
    - Therefore, $t^n > x$
    - By definition of $E = \{t \in \mathbb{R} | t > 0 \text{ and } t^n < x\}$

        - $t \notin E$ and t is greater than everything in E
        - Also $t \geq y - k$, so $y - k$ is an upper bound for E
        - But $y - k < y$, which contradicts $y = \sup E$
        - Thus, $y^n \ngtr x$
    - Therefore $y^n = x$

- Corollary: If $a, b \in \mathbb{R}^+$, and $n \in \mathbb{Z}^+$, then $a^{\frac{1}{n}} \cdot b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$
    - Let $\alpha = a^{\frac{1}{n}}, \beta = b^{\frac{1}{n}}$, then
    - $\alpha^n \beta^n = ab$
    - $(\alpha\beta)^n = ab$
    - So $\alpha\beta = (ab)^{\frac{1}{n}}$

Complex Numbers, Euclidean Spaces

Wednesday, February 7, 2018 12:12 PM

Complex Numbers

- Definition
  - If $z \in \mathbb{C}$, then $z = a + bi$ where $a, b \in \mathbb{R}$ and $i^2 = -1$
- Addition, multiplication, subtraction, and division
  - If $a + bi, c + di \in \mathbb{C}$, then
  - $(a + bi) + (c + di) = (a + c) + (b + d)i$
  - $(a + bi) - (c + di) = (a - c) + (b - d)i$
  - $(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$
  - $\dfrac{a + bi}{c + di} = \left(\dfrac{a + bi}{c + di}\right)\left(\dfrac{c - di}{c - di}\right) = \dfrac{(a + bi)(c - di)}{c^2 + d^2}$
- Real part and imaginary part
  - For $z = a + bi$
  - $\text{Re}(z) = a$ is the **real part** of z
  - $\text{Im}(z) = b$ is the **imaginary part** of z
- Complex conjugate
  - $\bar{z} = a - bi$ is the **complex conjugate** of z
  - $z\bar{z} = (a + bi)(a - bi) = a^2 + b^2$
- Absolute value
  - $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$ is the **absolute value** of z
  - Note
    - For a real number x
    - $|x| = \sqrt{x^2 + 0^2} = \sqrt{x^2} \geq 0$
    - $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$
- Complex division
  - If $z = a + bi, w = c + di \in \mathbb{C}$, then
  - $\dfrac{z}{w} = \dfrac{z\bar{w}}{w\bar{w}} = \dfrac{(a + bi)(c - di)}{(c + di)(c - di)} = \dfrac{ac + bd}{c^2 + d^2} + \dfrac{bc - ad}{c^2 + d^2}i$

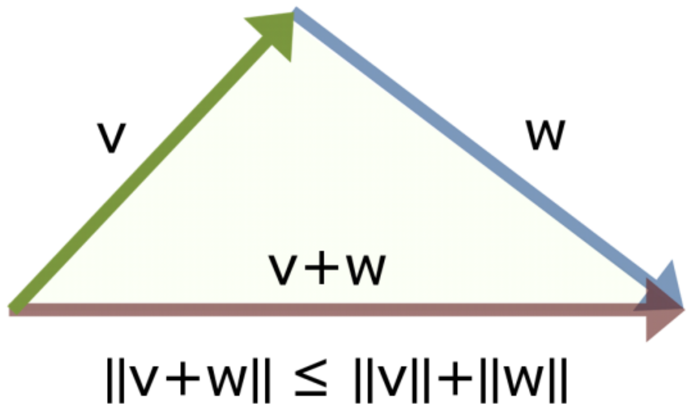
Theorem 1.31: Properties of Complex Numbers

- If z and w are complex numbers, then
- $\boldsymbol{\overline{z + w} = \bar{z} + \bar{w}}$
- $\boldsymbol{\overline{zw} = \bar{z} \cdot \bar{w}}$
- $\boldsymbol{z + \bar{z} = 2\text{Re}(z)}$

- $\boldsymbol{z - \bar{z} = 2i\,\mathrm{Im}(z)}$
- $\boldsymbol{z\bar{z}}$ **is real and positive** (except when $z = 0$)

Theorem 1.33: Properties of Complex Numbers

- If z and w are complex numbers, then
  (1) $\boldsymbol{|z| > 0}$ **unless** $\boldsymbol{z = 0}$ **in which case** $\boldsymbol{|z| = 0}$
  (2) $\boldsymbol{|\bar{z}| = |z|}$
  (3) $\boldsymbol{|zw| = |z||w|}$
    - Let $z = a + bi, w = c + di$
    - Then $zw = (ac - bd) + (ad + bc)i$
    - $|zw| = \sqrt{(ac - bd)^2 + (ad + bc)^2}$
    - $= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}$
    - $= \sqrt{(a^2 + b^2)(c^2 + d^2)}$
    - $= \sqrt{a^2 + b^2}\sqrt{c^2 + d^2}$
    - $= |z||w|$
  (4) $\boldsymbol{|\mathrm{Re}(z)| \leq |z|}$
  (5) $\boldsymbol{|z + w| \leq |z| + |w|}$ (Triangle Inequality)
    - $|z + w|^2 = (z + w)(\overline{z + w})$
    - $= (z + w)(\bar{z} + \bar{w})$
    - $= z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w}$
    - $= |z|^2 + |w|^2 + z\bar{w} + \bar{z}w$
    - $= |z|^2 + |w|^2 + 2\mathrm{Re}(z\bar{w})$
    - $\leq |z|^2 + |w|^2 + 2|z\bar{w}|$ by (4)
    - $= |z|^2 + |w|^2 + 2|z||\bar{w}|$ by (3)
    - $= |z|^2 + |w|^2 + 2|z||w|$ by (2)
    - $= (|z| + |w|)^2$
    - So $|z + w|^2 \leq (|z| + |w|)^2$
    - Thus, $|z + w| \leq |z| + |w|$



Definition 1.36: Euclidean Spaces

- Inner product
  - If $\vec{x}, \vec{y} \in \mathbb{R}^n$ with
    - $\vec{x} = (x_1, x_2, \dots, x_n)$
    - $\vec{y} = (y_1, y_2, \dots, y_n)$
  - Then the **inner product** of $\vec{x}$ and $\vec{y}$ is
    - $\vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i$
- Norm
  - If $\vec{x} \in \mathbb{R}^{\mathrm{n}}$, we define the **norm** of $\vec{x}$ to be $|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}}$
- Euclidean spaces
  - The vector space $\mathbb{R}^n$ with inner product and norm is called **Euclidean *n*-space**

Theorem 1.37: Properties of Euclidean Spaces

- Suppose $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n, \alpha \in \mathbb{R}$, then
- $|\vec{x}| \geq 0$
- $|\vec{x}| = 0$ if and only if $\vec{x} = \vec{0}$
- $|\alpha\vec{x}| = |\alpha| \cdot |\vec{x}|$
- $|\vec{x} \cdot \vec{y}| \leq |\vec{x}| \cdot |\vec{y}|$ (Schwarz's Inequality)
- $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$ (Triangle Inequality)
- $|\vec{x} - \vec{y}| \leq |\vec{x} - \vec{z}| + |\vec{y} - \vec{z}|$ (Triangle Inequality)

Theorem 1.35: Schwarz Inequality

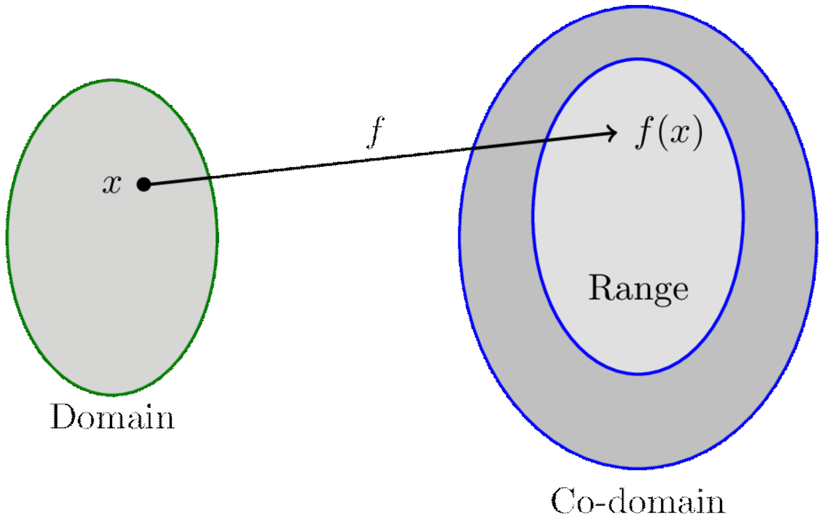
- Statement
  - $\left|\sum_{j=1}^{n} a_j \bar{b}_j\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$
- Proof
  - See Theorem 1.35 in Rudin for a proof of **Schwarz Inequality** for $\mathbb{C}$
  - For intuition, try proving $(x_1 y_2 + x_2 y_2)^2 \leq (x_1^2 + x_2^2)(y_1^2 + y_2^2)$
- **Triangle Inequality**
  - In a Euclidean Space, $|\vec{x} \cdot \vec{y}| \geq |\vec{x}| \cdot |\vec{y}|$
  - $|\vec{x} + \vec{y}|^2 = |\vec{x}|^2 + 2\vec{x} \cdot \vec{y} + |\vec{y}|^2 \leq |\vec{x}|^2 + 2|\vec{x}||\vec{y}| + |\vec{y}|^2 = (|\vec{x}| + |\vec{y}|)^2$
  - Thus $|\vec{x} + \vec{y}| < |\vec{x}| + |\vec{y}|$
  - Let $\vec{x} \coloneqq \vec{x} - \vec{y}, \vec{y} \coloneqq \vec{y} - \vec{z}$, we have $|\vec{x} - \vec{z}| < |\vec{x} - \vec{y}| + |\vec{y} - \vec{z}|$

Function, Cardinality, Equivalence Relation

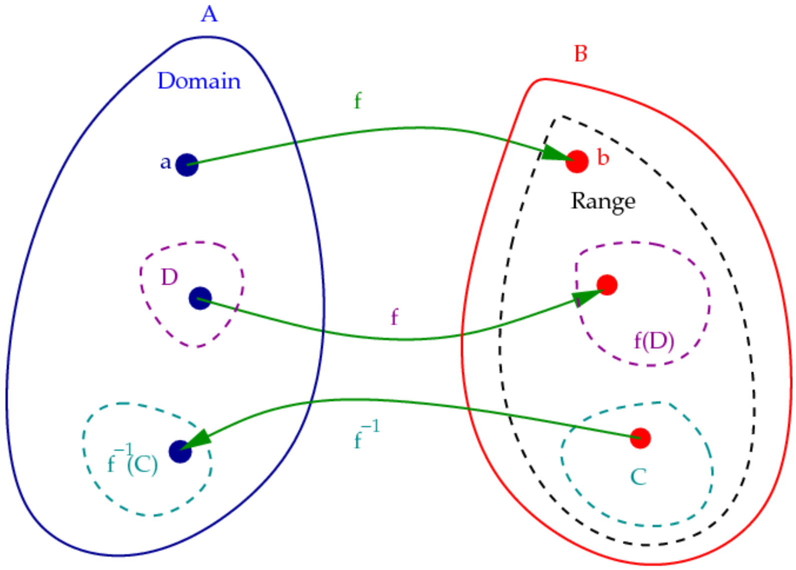
Monday, February 12, 2018 12:08 PM

Definition 2.1 & 2.2: Function

- Given two sets A and B
- A **function** (or **mapping**) is a rule that assigns elements in A to elements in B
- Notationally, if f is a function from A to B, we write $f: A \to B$



- Set A is called the **domain** of f
- Set B is called the **codomain** of f
- For $E \subset A$, $f(E) = \{b \in B | b = f(e) \text{ for some } e \in E\}$ is the **image** of E under f
- $f(A)$ is called the **range** of f
- If $f(A) = B$, then we say that f is **onto** or **surjective**
- If $f(a_1) = f(a_2)$ implies $a_1 = a_2$, then f is **one-to-one** or **injective**
- A function that is both one-to-one and onto is said to be **bijective**
- For $E \subset B$, $f^{-1}(E) = \{a \in A | f(a) \in E\}$ is the **inverse image** of E under f



- Notationally, if $y \in B$, $f^{-1}(y) = f^{-1}(\{y\})$
  - f^{-1} is at most a single element set for all $y \in B$ if and only if f is injective
  - In this case, f^{-1} can be thought of as a function maps to the single element
- Example
  - $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$
  - $f^{-1}(\{1\}) = \{1, -1\}$
  - $f^{-1}(\{x \in \mathbb{R} | x < 0\}) = \emptyset$
  - $f^{-1}(\{0\}) = \{0\}$, we can also write $f^{-1}(0) = 0$

Definition 2.3: Cardinality

- If there exists a one-to-one, onto mapping from set A to set B
- We say that A and B can be put in **one-to-one correspondence**
- And that A and B have the same **cardinality** (or **cardinal number**)
- In this case, we write $A \sim B$

Definition 2.3: Equivalence Relation

- One-to-one correspondence is an example of an equivalence relation
- An **equivalence relation** satisfies 3 properties
  - Reflexive: $A \sim A$
  - Symmetric: If $A \sim B$, then $B \sim A$
  - Transitivity: If $A \sim B, B \sim C$, then $A \sim C$

Cardinality and Countability, Sequence

Wednesday, February 14, 2018 12:06 PM

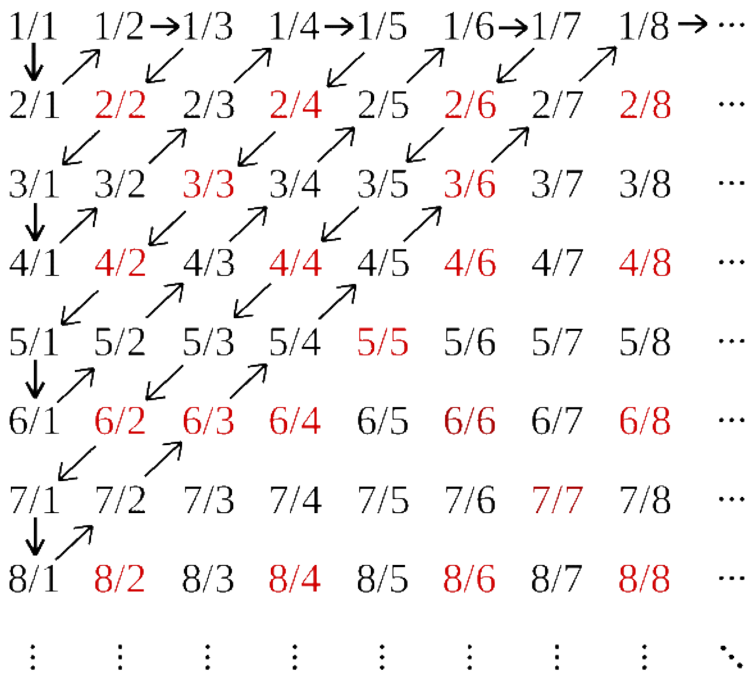
Definition 2.4: Cardinality and Countability

- Let $J_n = \{1,2,3,\dots,n\}$ and $\mathbb{N} = \{1,2,3,\dots\}$
- For any set A, we say
- A is **finite** if $A \sim J_n$ for some n ($\emptyset$ is also considered as finite)
- A is **infinite** if $A \nsim J_n$ for all n
- A is **countable** if $A \sim \mathbb{N}$
- A is **uncountable** if A is neither finite nor countable
- A is **at most coutable** if A is finite or countable

Examples 2.5: Countability

- $\mathbb{N}$ is countable
    - $\mathbb{N} = \{1,2,3,\dots\}$
- $\mathbb{Z}$ is countable
    - $\mathbb{Z} = \{0,1,-1,2,-2,3,-3,\dots\}$
    - Define $f: \mathbb{N} \to \mathbb{Z}$ by
    - $f(n) \coloneqq \begin{cases} \dfrac{n}{2} & n \text{ is even} \\ \dfrac{1-n}{2} & n \text{ is odd} \end{cases}$
    - f is injective
        - If $f(n) = f(m)$
        - then $\dfrac{n}{2} = \dfrac{m}{2}$ or $\dfrac{1-n}{2} = \dfrac{1-m}{2}$
        - Either way, $n = m$
    - f is surjective
        - Given $k \in \mathbb{Z}$,
        - If $k > 0, k = f(2k)$
        - If $k \leq 0, k = f(-2k+1)$
    - Thus f is bijective
- $\mathbb{Q}$ is countable
    - There are "less" rational numbers $q = \frac{m}{n}$ $(m, n \in \mathbb{Z}, n \neq 0)$ than
    - there are ordered pairs of integers (m, n)
        - $\dfrac{1}{2} = \dfrac{15}{30}$ but $(1,2) \neq (15,30)$
        - We can also ignore negatives and zeros

- because integers are in 1-1 correspondence with $\mathbb{N}$
- Idea: Write ordered pairs of integers in a 2 dimension array
- Putting this all together, we have
- $\mathbb{Q} = \left\{0, \pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm 3, \pm 4, \pm \frac{3}{2}, \pm \frac{2}{3}, \pm \frac{1}{4}, \dots\right\}$



Definition 2.7: Sequence

- Definition
  - A **sequence** is a function defined on $\mathbb{N}$
  - Notationally, this is often written $\{x_n\}$
  - Meaning $f(x) = x_n$ for all $n \in \mathbb{N}$
- Example
  - $\left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$

Theorem 2.8: Infinite Subset of Countable Set

- Statement
  - Every **infinite subset** of a **countable set** is **countable**
- Intuition
  - Countable sets represent the "smallest" infinity
  - No uncountable set can be a subset of a countable set.
- Proof
  - Let $E \subset A$
  - Suppose A is countable and E is infinite

- Since A is countable, its element will be a sequqnce
- (order given by the bijective function $f: \mathbb{N} \to A$)
- Let n_1 be the smallest $n \in \mathbb{N}$ such that $x_{n_1} \in E$
- Let n_2 be the next smallest $n \in \mathbb{N}$ such that $x_{n_2} \in E$
- So $E = \left\{x_{n_k}\right\} = \left\{x_{n_1}, x_{n_2}, x_{n_3}, \dots\right\}$
- i.e. E is a sequence indexed by $k \in \mathbb{N}$
- Now consider $g: \mathbb{N} \to E$ given by $g(k) = x_{n_k}$
- g is clearly one-to-one and onto by construction
- Therefore E is countable

- Example
  - Let $A \coloneqq \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$ and $E \coloneqq \left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots\right\}$
  - Then $A = \left\{\frac{1}{n}\right\}$, and $E = \left\{\frac{1}{n_k}\right\}$ where $n_k = k^2$ for $k \in \mathbb{N}$
  - Let $f: A \to E$ by $f(k) = \frac{1}{k^2}$
  - We can show that f is a bijection
  - Thus, E is countable

Set Operations, Countable and Uncountable

Friday, February 16, 2018 12:08 PM

Definition 2.9: Set-Theoretic Operations

- Set theoretic **union**
  - $\bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup A_3 \cup \cdots$
- Set theoretic **intersection**
  - $\bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap A_3 \cap \cdots$
- Indexing set
  - $\bigcup_{\alpha \in A} E_\alpha$, where
  - A is an **indexing set**
  - E_α is a specific set that depends on A
- Example
  - Let $A = \{x \in \mathbb{R} | 0 < x \leq 1\}$
  - Let $E_\alpha = \{x \in \mathbb{R} | 0 < x < a\}$
  - Then $\bigcup_{\alpha \in A} E_\alpha = (0,1)$ and $\bigcap_{\alpha \in A} E_\alpha = \emptyset$

Theorem 2.12: Union of Countable Sets

- Statement
  - Let $\{E_n\}_{n \in \mathbb{N}}$ be a sequence of **countable sets**, then
  - $S = \bigcup_{n=1}^{\infty} E_n$ is also **countable**
- Proof
  - Just like the proof that $\mathbb{Q}$ is countable
  - $E_n = \{x_{n_k}\} = \{x_{n_1}, x_{n_2}, x_{n_3}, \ldots\}$
  - $$\begin{matrix} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ x_{21} & x_{22} & x_{23} & \ddots & \\ x_{31} & x_{32} & \ddots & & \\ x_{41} & \ddots & & & \\ \vdots & & & & \end{matrix}$$
  - Go along the diagonal, we have
  - $S = \{x_{11}, x_{21}, x_{12}, x_{31}, x_{22}, x_{13} \ldots\}$
- Corollary

- Suppose A is **at most countable**
- If B_α is **at most countable** $\forall \alpha \in A$
- Then $\bigcup_{\alpha \in A} B_\alpha$ is also **at most countable**

Theorem 2.13: Cartesian Product of Countable Sets

- Statement
  - Let A be a **countable set**
  - Let B_n be the **set of all n-tuples** $(a_1, a_2, \ldots a_n)$ where
    - $a_k \in A$ for $k = 1,2, \ldots, n$
    - a_k may not be distinct
  - Then B_n is **countable**
- Proof
  - We proof by induction on n
  - Base case: $n = 2$
    - $$\begin{matrix} (a_1, a_1) & (a_1, a_2) & (a_1, a_3) & (a_1, a_4) & \cdots \\ (a_2, a_1) & (a_2, a_2) & (a_2, a_3) & \ddots & \\ (a_3, a_1) & (a_3, a_2) & \ddots & & \\ (a_4, a_1) & \ddots & & & \\ \vdots & & & & \end{matrix}$$
    - Here, a_i are all the elements of A with possible repetition
  - Now assume for $n = m$ where $m \geq 2$
    - The set of m-tuples $(a_1, a_2, \ldots a_m)$ are countable
    - Now we treat the $(m+1)$-tuples as ordered pairs
    - $(a_1, a_2, \ldots a_{m+1}) = \left((a_1, a_2, \ldots a_m), a_{m+1}\right)$
    - By $n = 2$ case, the set of $(m+1)$-tuples is still countable

Theorem 2.14: Cantor's Diagonalization Argument

- Statement
  - Let A be the **set of all sequqnecse** whose digits are 0 and 1
  - Then A is **uncountable**
- Proof: Cantor's Diagonalization Argument
  - Suppose A is countable
  - Then $A = \{s_1, s_2, s_3, \ldots\}$ where s_k is a sequence of 0 and 1 for all $k \in \mathbb{N}$
    - $s_1 = \{s_{11}, s_{12}, s_{13}, s_{14}, s_{15} \ldots\}$
    - $s_2 = \{s_{21}, s_{22}, s_{23}, s_{24}, s_{25} \ldots\}$
    - $s_3 = \{s_{31}, s_{32}, s_{33}, s_{34}, s_{35} \ldots\}$
    - $\vdots$

- where $s_{ij} \in \{0,1\}$ for $i, j \in \mathbb{N}$
- Construct a new sequence $s = \{x_1, x_2, x_3, \ldots\}$ where
    - $x_i = \begin{cases} 0 & \text{if } s_{ii} = 1 \\ 1 & \text{if } s_{ii} = 0 \end{cases}$
- Then $s \neq s_i, \forall i \in \mathbb{N}$
- So $S \notin A$, which is a contradiction
- Thus, A must be uncountable

- Corollary
    - $\mathbb{R}$ **is uncountable**

$$
\begin{array}{ll}
s_1 = & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ldots \\
s_2 = & 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \ldots \\
s_3 = & 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ \ldots \\
s_4 = & 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ \ldots \\
s_5 = & 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ \ldots \\
s_6 = & 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ \ldots \\
s_7 = & 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ \ldots \\
s_8 = & 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ \ldots \\
s_9 = & 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ \ldots \\
s_{10} = & 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ \ldots \\
s_{11} = & 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ \ldots \\
\vdots & \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \ddots
\end{array}
$$

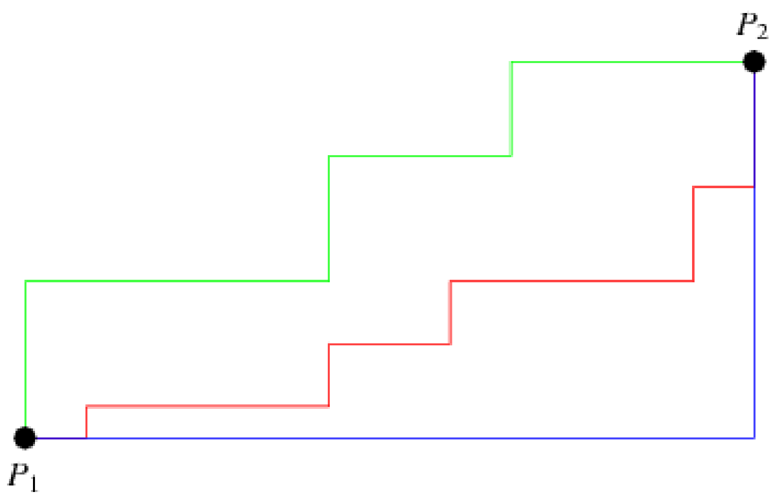
$$
s = 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ \ldots
$$

Metric Space, Interval, Cell, Ball, Convex

Monday, February 19, 2018 12:04 PM

Definition 2.15: Metric Space

- Definition
  - A set X of **points** is called a **metric space** if
  - there exists a metric or distance function $d(p,q): X \times X \to \mathbb{R}$ such that
    - **Positivity**
      - $d(p,q) > 0$ if $p,q \in X$ and $p \neq q$
      - $d(p,p) = 0$ for all $p \in X$
    - **Symmetry**
      - $d(p,q) = d(q,p)$ for all $p,q \in X$
    - **Triangle Inequality**
      - $d(p,q) \leq d(p,r) + d(r,q)$ for all $p,q,r \in X$
- Example 1
  - $X = \mathbb{R}^k$
  - $d(\vec{p},\vec{q}) = |\vec{p} - \vec{q}|$
  - If $k = 1$, this is just standard numerical absolute value
  - and d is the distance on the number line
- Example 2 (Taxicab metric)
  - $X = \mathbb{R}^2$
  - $d\left((p_1,p_2),(q_1,q_2)\right) = |p_1 - q_1| + |p_2 - q_2|$ where $p_1,p_2,q_1,q_2 \in \mathbb{R}$



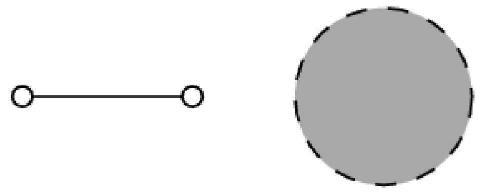
  - Is this a true metric space?
  - Positivity
    - Clearly $d\left((p_1,p_2),(q_1,q_2)\right) \geq 0$ since it is a sum of absolute values

- Suppose $d\left((p_1,p_2),(q_1,q_2)\right) = 0$
  - $|p_1 - q_1| + |p_2 - q_2| = 0$
  - $|p_1 - q_1| = -|p_2 - q_2|$
  - $\begin{cases}|p_1 - q_1| = 0 \\ |p_2 - q_2| = 0\end{cases} \Rightarrow \begin{cases}p_1 = q_1 \\ p_2 = q_2\end{cases}$
  - i.e. $(p_1,p_2) = (q_1,q_2)$
- Suppose $(p_1,p_2) = (q_1,q_2)$
  - $d\left((p_1,p_2),(q_1,q_2)\right) = |p_1 - q_1| + |p_2 - q_2| = |0| + |0| = 0$
- Thus $d\left((p_1,p_2),(q_1,q_2)\right) = 0 \Leftrightarrow (p_1,p_2) = (q_1,q_2)$

- Symmetry
  - $d\left((p_1,p_2),(q_1,q_2)\right) = |p_1 - q_1| + |p_2 - q_2|$
  - $= |q_1 - p_1| + |q_2 - p_2| = d\left((q_1,q_2),(p_1,p_2)\right)$
- Triangular Inequality
  - $d\left((p_1,p_2),(r_1,r_2)\right) + d\left((r_1,r_2),(q_1,q_2)\right)$
  - $= |p_1 - r_1| + |p_2 - r_2| + |r_1 - q_1| + |r_2 - q_2|$
  - $= \left(|p_1 - r_1| + |r_1 - q_1|\right) + \left(|p_2 - r_2| + |r_2 - q_2|\right)$
  - $\geq |p_1 - r_2 + r_1 - q_1| + |p_2 - r_2 + r_2 - q_2|$ by Triangle Inequality of $\mathbb{R}$
  - $= |p_1 - q_1| + |p_2 - q_2|$
  - $= d\left((p_1,p_2),(q_1,q_2)\right)$

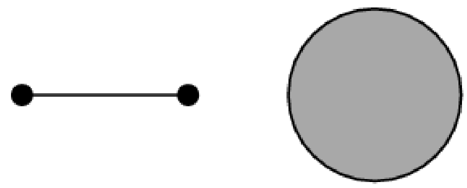
Definition 2.17: Interval, k-cell, Ball, Convex

- Interval
  - **Segment** (a,b) is $\{x \in \mathbb{R} | a < x < b\}$ (**open interval**)
  - **Interval** $[a,b]$ is $\{x \in \mathbb{R} | a \leq x \leq b\}$ (**closed interval**)
  - We can also have **half-open intervals**: $(a,b]$ and $[a,b)$
- k-cell
  - If $a_i < b_i$ for $i = 1,2,\dots,k$
  - The set of points $\vec{x} = (x_1,x_2,\dots,x_k)$ in $\mathbb{R}^k$
  - that satisfy $a_i \leq x_i \leq b_i$ $(1 \leq i \leq k)$ is called a ***k*-cell**
- Ball
  - If $\vec{x} \in \mathbb{R}^k$ and $r > 0$
  - The **open ball** with center $\vec{x}$ with radius r is $\left\{\vec{y} \in \mathbb{R}^k \middle| |\vec{x} - \vec{y}| < r\right\}$



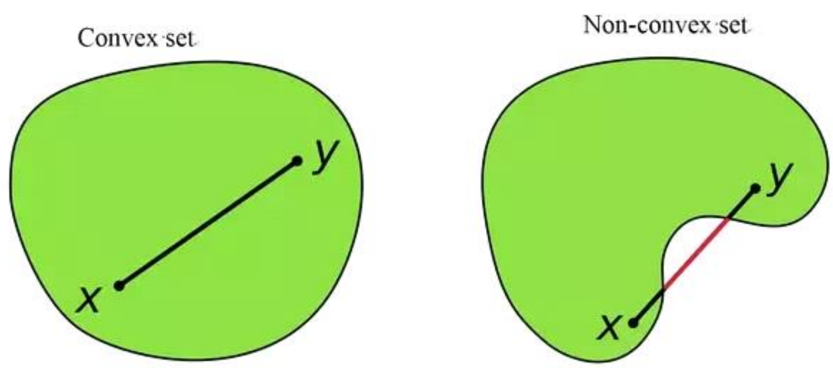
open interval *open disk*

- the **closed ball** with center $\vec{x}$ with radius r is $\{\vec{y} \in \mathbb{R}^k \mid |\vec{x} - \vec{y}| \le r\}$



closed interval *closed disk*

- Convex
  - We call a set $E \subset \mathbb{R}^k$ **convex** if
  - $\lambda\vec{x} + (1-\lambda)\vec{y} \in E, \forall \vec{x}, \vec{y} \in E, 0 < \lambda < 1$
  - i.e. All points along a straight line from $\vec{x}$ to $\vec{y}$ and between $\vec{x}$ and $\vec{y}$ is in E



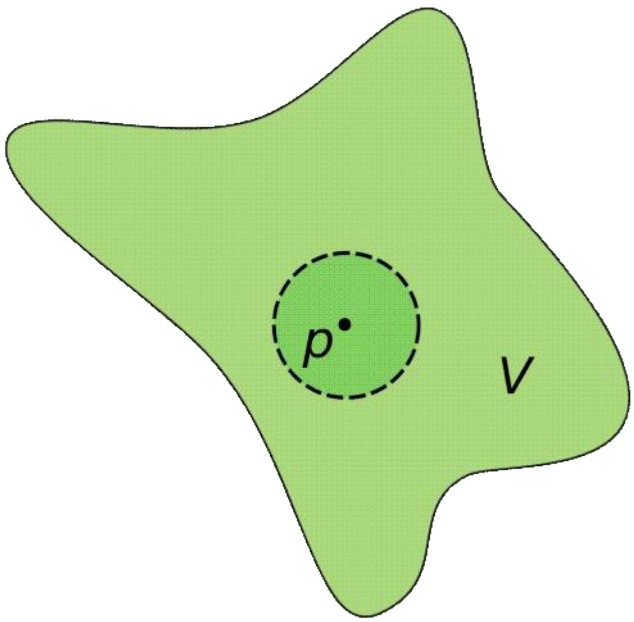
- Example: Balls are convex
  - Given an open ball with center $\vec{x}$ and radius r
  - If $\vec{y}, \vec{z} \in B$, then $|\vec{y} - \vec{x}| < r$ and $|\vec{z} - \vec{x}| < r$
  - $|\lambda\vec{z} + (1-\lambda)\vec{y} - \vec{x}|$
  - $= |\lambda\vec{z} + (1-\lambda)\vec{y} - (\lambda + 1 - \lambda)\vec{x}|$
  - $= |\lambda\vec{z} - \lambda\vec{x} + (1-\lambda)\vec{y} - (1-\lambda)\vec{x}|$
  - $\le |\lambda\vec{z} - \lambda\vec{x}| + |(1-\lambda)\vec{y} - (1-\lambda)\vec{x}|$ by Triangle Inequality
  - $= \lambda|\vec{z} - \vec{x}| + (1-\lambda)|\vec{y} - \vec{x}|$
  - $< \lambda r + (1-\lambda)r = r$
  - Thus $|\lambda\vec{z} + (1-\lambda)\vec{y} - \vec{x}| < r$
  - i.e. $\lambda\vec{z} + (1-\lambda)\vec{y} \in B$

Definitions in Metric Space

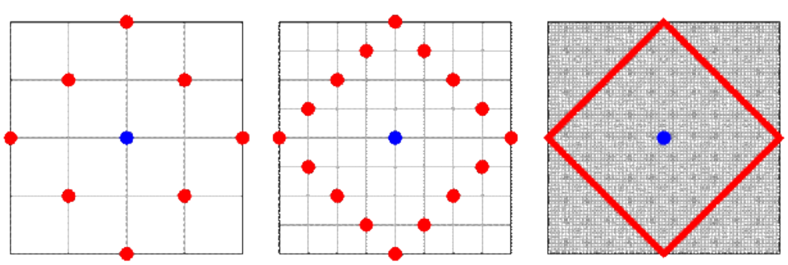
Wednesday, February 21, 2018 12:01 PM

Definitions 2.18: Definitions in Metric Space

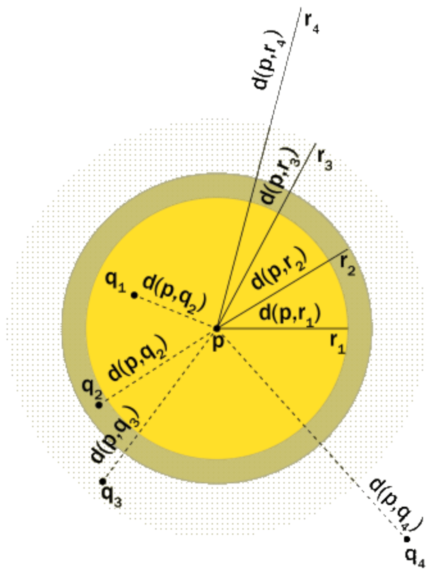
- Let X be a metric space. All points/elements below are in X
- Neighborhood
    - Definition
        - A **neighborhood** of p is a set $N_r(p)$ consisting of
        - all points q such that $d(p,q) < r$ for some $r \in \mathbb{R}$
        - We call r the radius of $N_r(p)$
    - Example: $\mathbb{R}^2$



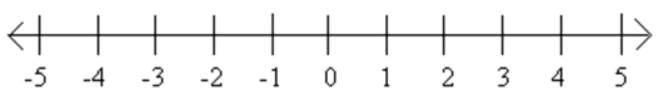
    - Example: Taxicab metric



- Limit point
    - Definition
        - A point p is a **limit point** of the set $E \subset X$ if
        - every neighborhood of p contains a point $q \in E$ and $p \neq q$
    - Example: $\mathbb{R}^2$



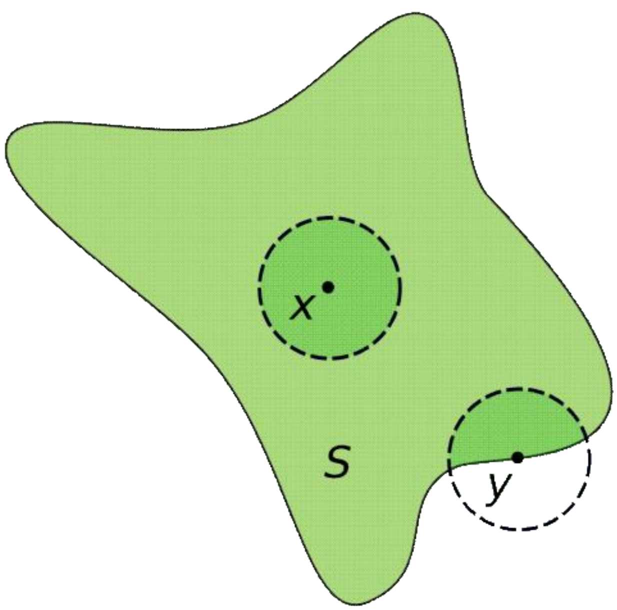
- Example: $(0,1) \in \mathbb{R}$
  - For $(0,1) \in \mathbb{R}$, the limit points is $[0,1]$
- Isolated point
  - Definition
    - If $p \in E$ and p is not a limit point of E, then
    - p is an **isolated point** of E
  - Example: $\mathbb{Z}$ in $\mathbb{R}$
    - Every integers is an isolated point in $\mathbb{R}$



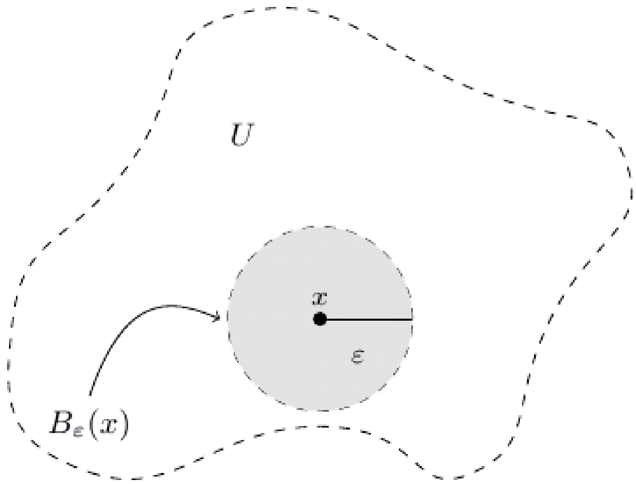
- Closed set
  - Definition
    - A set E is **closed** if every limit point of E is in E
  - Example: $[0,1] \in \mathbb{R}$
    - In $\mathbb{R}$, neighborhood of $p \in \mathbb{R}$ are open intevals cenerted about p
    - All of $[0,1]$ is a limit point since
    - If $x \in [0,1]$
      - The neighborhood about x is $(x - r, x + r)$
      - $(x - r, x + r) \cap [0,1]$ is non-empty
      - If $x = 0$, then take $q = \min\left(x + \frac{r}{2}, 1\right)$
      - Otherwise take $q = \max\left(x - \frac{r}{2}, 0\right)$
      - So every point in $[0,1]$ is a limit point
    - If $x \notin [0,1]$
      - i.e. $x < 0$ or $x > 1$

- Take $r = \begin{cases} |x| & \text{if } x < 0 \\ |x-1| & \text{if } x > 1 \end{cases}$
- Then $N_r(x) \cap [0,1] = \emptyset$
- So nothing outside of $[0,1]$ is a limit point of $[0,1]$
- So $[0,1]$ contains all its limit points
- Thus $[0,1]$ is closed

- Interior point
  - Definition
    - A point p is an **interior point** of a set E if
    - there exists a neighborhood $N_r(p)$ that is a subset of E
  - Example: $\mathbb{R}^2$
    - For the closed set S
    - The point x is an interior point of S
    - The point y is not an interior point of S (on the boundary of S)



- Open set
  - Definition
    - E is an **open set** if every point of E is an interior point
  - Example: $\mathbb{R}^2$
    - U is an open set, since $\forall x \in U, \exists B_\epsilon(x) \subset U$



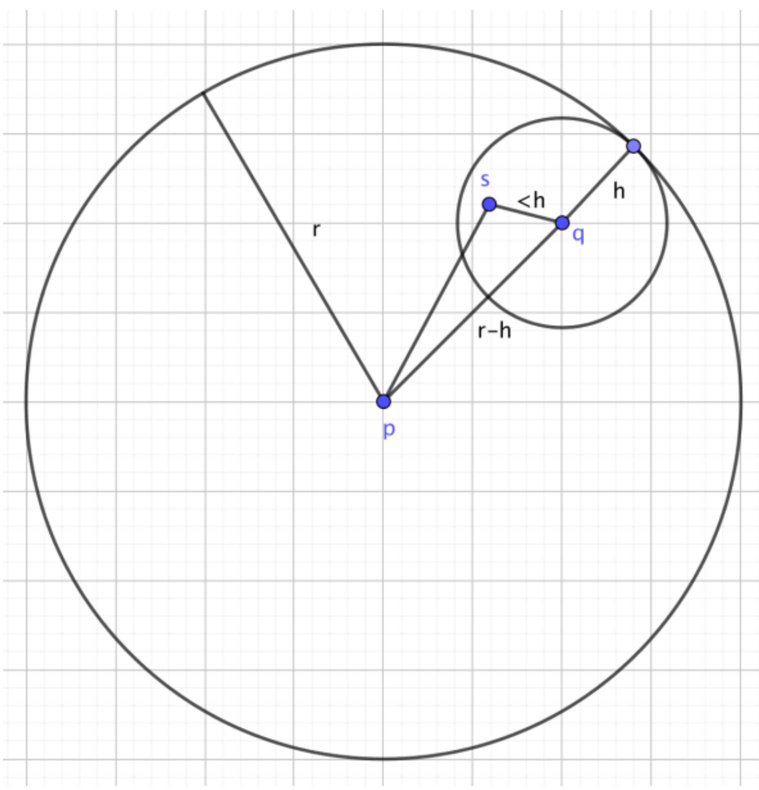
- Example: $(0,1) \in \mathbb{R}$
  - For $x \in (0,1)$
  - Take $r = \min(x, 1 - x)$
  - $N_r(x) \subset (0,1)$
  - Thus every point in $(0,1)$ is an interior point
- Complement
  - The **complement** of E (denoted as E^c) is $\{p \in X | p \notin E\}$
- Perfect
  - E is **perfect** if E is closed and every point of E is limit point of E
- Bounded
  - E is **bounded** if there is a real number M and a point $p \in E$ s.t.
  - $d(p, q) < M$ for all $p \in E$
- Dense
  - E is **dense** in X if
  - every point of X is a limit point of E or a point of E (or both)

Neighborhood, Open and Closed, De Morgan's Law

Friday, February 23, 2018 12:06 PM

Theorem 2.19: Every Neighborhood is an Open Set

- Statement
  - Every **neighborhood** is an **open set**
- Proof
  - Let X be a metric space
  - Choose a neighborhood $N_r(p) \subset X$
  - Let $q \in N_r(p)$
  - Choose $h \in \mathbb{R}$ s.t. $d(p, q) = r - h$
  - Consider the neighborhood $N_h(q)$
  - Let $s \in N_h(q)$, then $d(q, s) < h$
  - $d(p, s) \leq d(p, q) + d(q, s) < r - h + h = r$
  - Thus $d(p, s) < r$
  - i.e. $s \in N_r(p)$
  - So $N_h(q) \subset N_r(p)$
  - Therefore $N_r(p)$ is open

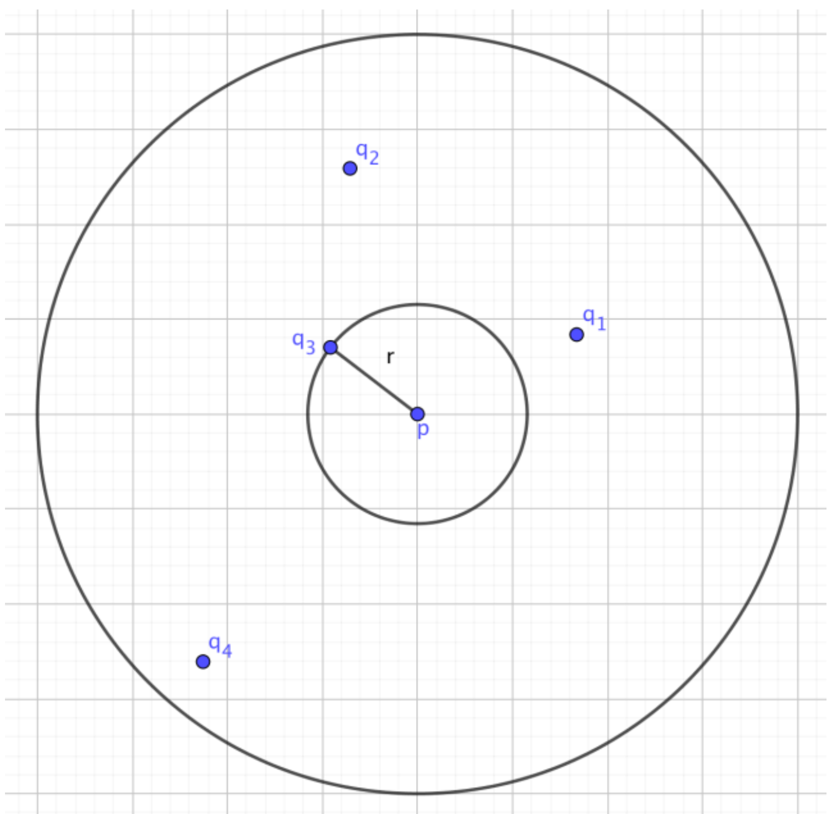


Theorem 2.20: Property of Limit Point

- Statement

- If p is a limit point of E
- Then **every neighborhood** of p contains **infinitely many points** of E

- Proof
  - Suppose the opposite
  - Then there exists a set E with a limit point p s.t.
  - The neighborhood of p contains only finitely many points of E
  - Namely $q_1, q_2, \dots, q_n$
  - Let $r = \min\big(d(p, q_1), d(p, q_2), \dots, d(p, q_n)\big)$
  - By definition, $q_i \notin N_r(p)$ for $1 \leq i \leq n$
  - This contradicts the fact that p is a limit point
  - So, this neighborhood about p must contain infinitely many points
- Corollary
  - A **finite set** has **no limit points**



Theorem 2.22: De Morgan's Law

- Statement
  - Let $\{E_\alpha\}$ be a finite or infinite collection of sets, then
  - $\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha)^c$
- Proof ($\Longrightarrow$)

- Suppose $x \in \left(\bigcup_{\alpha} E_{\alpha}\right)^{c}$
- Then $x \notin \bigcup_{\alpha} E_{\alpha}$
- So $x \notin E_{\alpha}, \forall \alpha$
- Thus, $x \in (E_{\alpha})^{c}$ for all α
- So, $x \in \bigcap_{\alpha} (E_{\alpha})^{c}$
- i. e. $\left(\bigcup_{\alpha} E_{\alpha}\right)^{c} \subset \bigcap_{\alpha} (E_{\alpha})^{c}$

- Proof ($\Longleftarrow$)
  - Suppose $x \in \bigcap_{\alpha} (E_{\alpha})^{c}$
  - Then $x \in (E_{\alpha})^{c}$ for all α
  - So $x \notin E_{\alpha}$ for all α
  - $x \notin \bigcup_{\alpha} E_{\alpha}$
  - Thus, $x \in \left(\bigcup_{\alpha} E_{\alpha}\right)^{c}$
  - i. e. $\bigcap_{\alpha} (E_{\alpha})^{c} \subset \left(\bigcup_{\alpha} E_{\alpha}\right)^{c}$

Theorem 2.23: Complement of Open/Closed Set

- Statement
  - A set E is **open** if and only if E^{c} is **closed**
  - Note: This does not say that open is not closed and closed is not open
- Proof ($\Longleftarrow$)
  - Suppose E^{c} is closed
  - Choose $x \in E$, so $x \notin E^{c}$
  - So, x is not a limit point of E^{c}
  - i.e. There exists a neighborhood $N_{r}(x)$ that contains no points of E^{c}
  - So, $N_{r}(x) \cap E^{c} = \emptyset$
  - Consequently, $N_{r}(x) \subset E$
  - So, x is an interior point of E
  - By definition, E is open

- Proof ($\Longrightarrow$)
  - Suppose E is open
  - Let x be a limit point of E^c (if exists)
  - So, every neighborhood of x contains a point in E^c
  - So, x is not an interior point of E
  - E is open, so $x \in E^c$
  - Thus, E^c contains its limit points and is closed by definition
- Corollary
  - A set E is **closed** if and only if E^c is **open**

Examples 2.21: Closed, Open, Perfect and Bounded

- Let $X = \mathbb{R}^2$

Subset	Closed	Open	Perfect	Bounded
$\{\vec{x} \in \mathbb{R}^2 \big\| \|\vec{x}\| < 1\}$	×	✓	×	✓
$\{\vec{x} \in \mathbb{R}^2 \big\| \|\vec{x}\| \leq 1\}$	✓	×	✓	✓
A nonempty finite set	✓	×	×	✓
$\mathbb{Z}$	✓	×	×	×
$\{1/n \big\| n \in \mathbb{N}\}$	×	×	×	✓
$\mathbb{R}^2$	✓	✓	✓	×
(a, b)	×	?	×	✓

- Note: (a, b) is open as a subset of $\mathbb{R}$, but not as a subtset of $\mathbb{R}^2$

Open and Closed, Closure

Monday, February 26, 2018 12:06 PM

Theorem 2.24: Intersection and Union of Open/Closed Sets

(a) For **any** collection $\{G_n\}$ of **open** sets, $\bigcup_\alpha G_\alpha$ is **open**

- Suppose G_α is open for all α
- Let $G = \bigcup_\alpha G_\alpha$
- If $x \in G$, then $x \in G_\alpha$ for some α
- Since G_α is open, there is a neighborhood about x in G_α
- And consequently, the neighborhood about x is also in G
- Thus G is open

(b) For **any** collection $\{F_n\}$ of **closed** sets, $\bigcap_\alpha F_\alpha$ is **closed**

- Suppose F_α is closed for all α
- Then F_α^c is open by Theorem 2.23
- So $\bigcup_\alpha F_\alpha^c$ is open by (a)
- $\left(\bigcap_\alpha F_\alpha\right)^c = \bigcup_\alpha F_\alpha^c$, by De Morgan's Law
- Thus, $\left(\bigcap_\alpha F_\alpha\right)^c$ is open
- Therefore $\bigcap_\alpha F_\alpha$ is closed by Theorem 2.23

(c) For any **finite** collection, $G_1, G_2, \dots, G_n$ of **open** sets, $\bigcap_{i=1}^{n} G_i$ is also **open**

- Suppose $G_1, G_2, \dots, G_n$ is open
- Let $x \in H = \bigcap_{i=1}^{n} G_i$
- So, $x \in G_i$ for $1 \leq i \leq n$
- By definition, since each G_i is open
- x is contained in a neighborhood $N_{r_i}(x) \subset G_i$
- Let $r = \min(r_1, r_2, \dots, r_n)$

- $N_r(x) \subset G_i$ for $1 \leq i \leq n$
- So, $N_r(x) \in H$
- Thus, $H = \bigcap_{i=1}^{n} G_i$ is open

(d) For any **finite** collection, $F_1, F_2, \dots, F_n$ of **closed** sets, $\bigcup_{i=1}^{n} F_i$ is also **closed**

- Suppose $F_1, F_2, \dots, F_n$ is closed
- Then F_i^c is open by Theorem 2.23
- So $\bigcap_{i=1}^{n} F_i^c$ is open by (c)
- $\left(\bigcup_{i=1}^{n} F_i\right)^c = \bigcap_{i=1}^{n} F_i^c$, by De Morgan's Law
- Thus, $\left(\bigcup_{i=1}^{n} F_i\right)^c$ is open
- Therefore $\bigcup_{i=1}^{n} F_i$ is closed by Theorem 2.23

- Note
  - $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$
  - $\left(-\frac{1}{n}, \frac{1}{n}\right)$ is open $\forall n \in \mathbb{N}$, while $\{0\}$ is closed

Definition 2.26: Closure

- Let X be a metric space
- If $E \subset X$ and E' denotes the set of limit points of E in X
- Then the **closure** of E is defined to be $\bar{E} = E \cup E'$

Theorem 2.27: Properties of Closure

- If X is a metric space and $E \subset X$, then
- $\bar{E}$ is closed
  - Let $p \in \bar{E}^c$
  - Then p is neither a point of E nor a limit point of E
  - So there exists a neighborhood N about p that contains no points of E
  - So, $N \subset \bar{E}^c$
  - i.e. every point of $\bar{E}^c$ is an interior point
  - Thus $\bar{E}^c$ is open

    - Therefore $\bar{E}$ is closed
- $E = \bar{E} \Leftrightarrow E$ is closed
    - If $E = \bar{E}$, then E is closed
    - If E is closed, E contains its limit points, so $E' \subset E$ and $E = \bar{E}$
- $\bar{E} \subset F$ for every closed set $F \subset X$ s.t. $E \subset F$
    - Suppose F is closed and $E \subset F$
    - F is closed $\Rightarrow F' \subset F$
    - $E \subset F \Rightarrow E' \subset F' \subset F$
    - Thus $\bar{E} = E \cup \mathrm{E}' \subset \mathrm{F}$
- Intuition: $\bar{E}$ is the smallest closed set in X containing E

Theorem 2.28: Closure and Least Upper Bound Property of $\mathbb{R}$

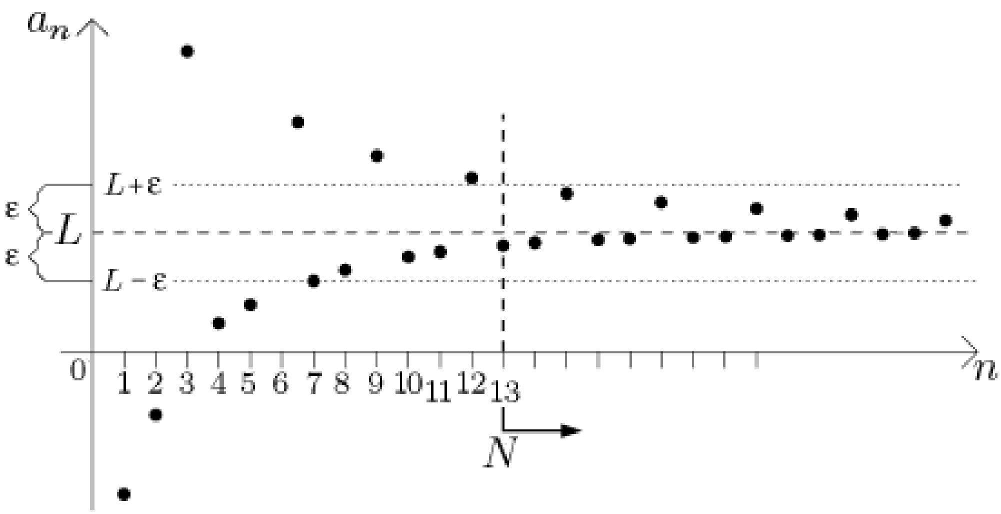
- Statement
    - If $E \neq \emptyset, E \subset \mathbb{R}$, and E is bouned above, then $\sup E \in \bar{E}$
    - Hence $\boldsymbol{\sup E \in E}$ **if** $\boldsymbol{E}$ **is closed**
- Proof
    - Let $y = \sup E$
    - If $y \in E$
        - Clearly $y \in \bar{E}$
    - If $y \notin E$
        - Let $h > 0$
        - Let $x \in (y - h, y)$
        - Suppose $\nexists x \in E$, then $y - h$ is an upper bound for E
        - But this contradicts the fact that $y = \sup E$
        - So there must be some $x \in E$ with $y - h < x < y$
        - Thus, for any neighborhood about y, $\exists x \in E$ in the neighborhood
        - So y is a limit point of E
        - i.e. $y \in E' \subset \bar{E}$

Convergence and Divergence, Range, Boundedness

Wednesday, February 28, 2018 12:07 PM

Definition 3.1: Convergence and Divergence

- Definition
  - A sequence $\{p_n\}$ in a metric space X **converges** to a point $p \in X$ if
  - Given any $\varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $d(p, p_n) < \varepsilon, \forall n \geq N$
  - If $\{p_n\}$ converges to p, we write
    - $p_n \to p$
    - $\lim_{n\to\infty} p_n = p$
    - $\lim p_n = p$
  - If $\{p_n\}$ does not converge, it is said to **diverge**
- Intuition
  - ε is small
  - N is a "point of no return" beyond which sequence is within ε of p



Range

- Given a sequence $\{p_n\}$
- The set of points p_n $(n \in \mathbb{N})$ is called the **range** of the sequence
- Range could be infinite, but it is always **at most countable**
- Since we can always construct a function $f: \mathbb{N} \to \{p_n\}$, where $f(n) = p_n$

Boundedness

- A sequence $\{p_n\}$ is said to be **bounded** if its range is bounded

Examples of Limit, Range and Boundedness

- Consider the following sequences of complex numbers

$\{s_n\}$	Limit	Range	Bounded
$s_n = \frac{1}{n}$	0	Infinite	Yes
$s_n = n^2$	Divergent	Infinite	No
$s_n = 1 + \frac{(-1)^n}{n}$	1	Infinite	Yes
$s_n = i^n$	Divergent	$\{\pm 1, \pm i\}$	Yes
$s_n = 1$	1	$\{1\}$	Yes

- Proof: $\lim_{n\to\infty} \frac{1}{n} = 0$
  - Let $\varepsilon > 0$
  - By Archimedean Property, we can choose $N \in \mathbb{N}$ s.t. $N > \frac{1}{\varepsilon}$
  - $\forall n \geq N, n > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n} < \varepsilon$
  - i.e. $d\left(\frac{1}{n}, 0\right) = \left|\frac{1}{n}\right| < \varepsilon, \forall n \geq N$
  - Therefore $\lim_{n\to\infty} \frac{1}{n} = 0$

Important Properties of Convergent Sequences

Friday, March 2, 2018 12:06 PM

Theorem 3.2: Important Properties of Convergent Sequences

- Let $\{p_n\}$ be a sequence in a metric space X
- $\boldsymbol{p_n \to p} \in X \Leftrightarrow$ any **neighborhood** of p contains p_n for **all but finitely many n**
  - Suppose $\{p_n\}$ converges to p
    - Let B be a neighborhood of p with radius ε
    - $p_n \to p \Rightarrow \exists N \in \mathbb{N} \text{ s.t. } d(p_n, p) < \varepsilon, \forall n \geq N$
    - So, $p_n \in B, \forall n \geq N$
    - $p_1, \dots, p_{n-1}$ may not be in B, but there are only finitely many of these
  - Suppose every neighborhood of p contains all but finitely many p_n
    - Let $\varepsilon > 0$ be given
    - $B \coloneqq \{q \in X | d(p,q) < \varepsilon\}$ is a neighborhood of p
    - By assumption, all but finitely points in $\{p_n\}$ are in B
    - Choose $N \in \mathbb{N}$ s.t. $N > i, \forall p_i \notin B$
    - Then $d(p_n, p) < \varepsilon, \forall n \geq N$
    - So, $\lim_{n\to\infty} p_n = p$
- Given $p \in X$ and $p' \in X$. If $\{\boldsymbol{p_n}\}$ **converges to p and to p'**, then $\boldsymbol{p = p'}$
  - Let $\varepsilon > 0$ be given
    - $\{p_n\}$ converges to $p \Rightarrow \exists N \in N \text{ s.t. } d(p_n, p) < \frac{\varepsilon}{2}, \forall n \geq N_1$
    - $\{p_n\}$ converges to $p' \Rightarrow \exists N' \in N \text{ s.t. } d(p_n, p') < \frac{\varepsilon}{2}, \forall n \geq N_2$
  - Let $N = \max(N_1, n_2)$, then
    - $d(p, p') \leq d(p_n, p) + d(p_n, p') < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \forall n \geq N$
  - Since $\varepsilon > 0$ is arbitrary, $d(p, p') = 0$
  - Therefore $p = p'$
- If $\{p_n\}$ **converges**, then $\{p_n\}$ is **bounded**
  - Since $\{p_n\}$ converges to some p
  - Let $\varepsilon = 1$, then $\exists N \in \mathbb{N}$ s.t. $d(p_n, p) < 1$
  - Let $q = \max\left(1, d(p_1, p), d(p_2, p), \dots, d(p_{N-1}, p)\right)$
  - Then $d(p, p_n) < q, \forall n \in \mathbb{N}$
  - By definition, $\{p_n\}$ is bounded
- If $E \subset X$, and $\boldsymbol{p \in E'}$, then there **exists a sequence** $\{\boldsymbol{p_n}\}$ **in E** s.t. $\boldsymbol{p_n \to p}$

- Since p is a limit point of E
- Every neighborhood of p contains $q \neq p$, and $q \in E$
- Consequently, $\forall n \in N, \exists p_n \in E$ s.t. $d(p_n, p) < \frac{1}{n}$
- Let $\varepsilon > 0$ be given
- By Archimedean property, $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$
- So for $n \geq N, \frac{1}{n} < \varepsilon \Rightarrow d(p_n, p) < \frac{1}{n} < \varepsilon$
- Therefore $p_n \to p$

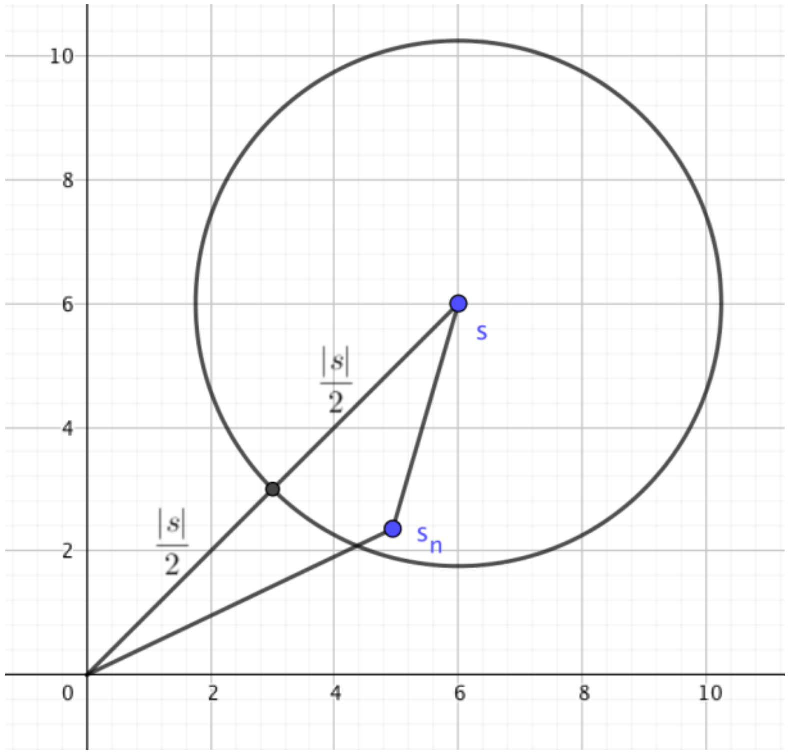
Algebraic Limit Theorem

Monday, March 5, 2018 12:10 PM

Theorem 3.3: Algebraic Limit Theorem

- Suppose $\{s_n\}, \{t_n\}$ are complex sequence, and $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$, then
- $\boldsymbol{\lim_{n\to\infty} s_n + t_n = s + t}$
  - Given $\varepsilon > 0$
    - $\lim_{n\to\infty} s_n = s \Rightarrow \exists N_1 \in \mathbb{N}\ \text{s.t.} \left|s_n - s\right| < \frac{\varepsilon}{2}$ for $n \geq N_1$
    - $\lim_{n\to\infty} t_n = t \Rightarrow \exists N_2 \in \mathbb{N}\ \text{s.t.} \left|t_n - t\right| < \frac{\varepsilon}{2}$ for $n \geq N_2$
  - Let $N = \max(N_1, N_2)$, then for $n \geq N$
    - $\left|s_n + t_n - (s + t)\right| = \left|(s_n - s) + (t_n - t)\right| \leq \left|s_n - s\right| + \left|t_n - t\right| < \varepsilon$
  - Therefore $\lim_{n\to\infty} s_n + t_n = s + t$
- $\boldsymbol{\lim_{n\to\infty} c + s_n = c + s}, \forall c \in \mathbb{C}$
  - Given $\varepsilon > 0$
  - $\lim_{n\to\infty} s_n = s \Rightarrow \exists N \in \mathbb{N}\ \text{s.t.} \left|s_n - s\right| < \varepsilon$ for $n \geq N$
  - So, $\left|c + s_n - (c + s)\right| = \left|s_n - s\right| < \varepsilon$
  - Therefore $\lim_{n\to\infty} c + s_n = c + s$
- $\boldsymbol{\lim_{n\to\infty} cs_n = cs}, \forall c \in \mathbb{C}$
  - Given $\varepsilon > 0$
  - If $c = 0$
    - $\left|cs_n - cs\right| = 0 < \varepsilon$
  - If $c \neq 0$
    - $\lim_{n\to\infty} s_n = s \Rightarrow \exists N \in \mathbb{N}\ \text{s.t.} \left|s_n - s\right| < \frac{\varepsilon}{|c|}$ for $n \geq N$
    - So $\left|cs_n - cs\right| = |c|\left|s_n - s\right| < |c|\frac{\varepsilon}{|c|} = \varepsilon$
  - Therefore $\lim_{n\to\infty} cs_n = cs$
- $\boldsymbol{\lim_{n\to\infty} s_n t_n = st}$
  - Standard approach
    - $s_n t_n - st = s_n t_n - st_n + st_n - st = t_n(s_n - s) + s(t_n - t)$
  - Rudin's approach
    - $s_n t_n - st = (s_n - s)(t_n - t) + t(s_n - s) + s(t_n - t)$
  - Given $\varepsilon > 0$

        - $\exists N_1 \in \mathbb{N}$ s.t. $|s_n - s| < \sqrt{\varepsilon}$ for $n \geq N_1$
        - $\exists N_2 \in \mathbb{N}$ s.t. $|t_n - t| < \sqrt{\varepsilon}$ for $n \geq N_2$
    - Let $N = \max(N_1, N_2)$, then
        - $|(s_n - s)(t_n - t)| < \varepsilon$ for $n \geq N$
        - $\Rightarrow \lim_{n\to\infty}(s_n - s)(t_n - t) = 0$
    - $\lim_{n\to\infty} s_n t_n$
        - $= \lim_{n\to\infty}[(s_n - s)(t_n - t) + t(s_n - s) + s(t_n - t) + st]$
        - $= \lim_{n\to\infty}(s_n - s)(t_n - t) + t\lim_{n\to\infty}(s_n - s) + s\lim_{n\to\infty}(t_n - t) + st$
        - $= 0 + 0 + 0 + st$
        - $= st$
    - Therefore $\lim_{n\to\infty} s_n t_n = st$
- $\boldsymbol{\lim_{n\to\infty}\frac{1}{s_n} = \frac{1}{s}}$ $(s_n \neq 0, \forall n \in \mathbb{N}, \text{and } s \neq 0)$
    - $\lim_{n\to\infty} s_n = s \Rightarrow \exists N' \in \mathbb{N}\text{ s.t. } |s_n - s| < \frac{|s|}{2}$ for $n \geq N'$
    - By the Triangle Inequality, $|s| - |s_n| \leq |s_n - s|, \forall n \geq N'$
    - $\Rightarrow |s_n| \geq |s| - |s_n - s| > |s| - \frac{|s|}{2} = \frac{|s|}{2}, \forall n \geq N'$
    - Given $\varepsilon > 0, \exists N > N'\text{ s.t. } |s_n - s| < \frac{1}{2}|s|^2\varepsilon$ for $n \geq N$
    - $\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{s_n s}\right| < \frac{\frac{1}{2}|s|^2\varepsilon}{|s_n| \cdot |s|} < \frac{\frac{1}{2}|s|^2\varepsilon}{\frac{|s|}{2} \cdot |s|} = \varepsilon$
    - Therefore $\lim_{n\to\infty}\frac{1}{s_n} = \frac{1}{s}$



10
8
6
4
2
0
2
4
6
8
10
s
s_n
$\frac{|s|}{2}$
$\frac{|s|}{2}$

Sequence Convergence in $\mathbb{R}^n$, Compact Set

Wednesday, March 7, 2018 12:15 PM

Theorem 3.4: Convergence of Sequence in $\mathbb{R}^n$

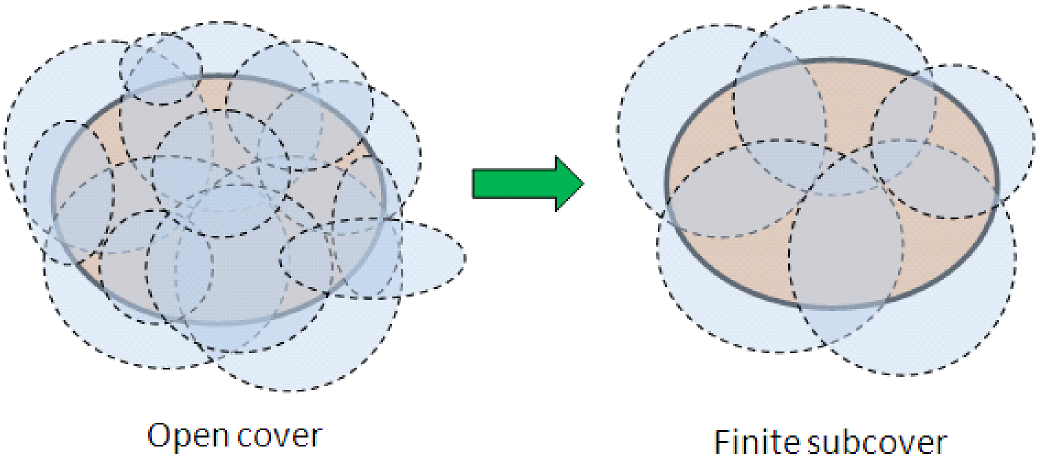
- Statement (a)
  - Suppose $\overrightarrow{x_n} = (\alpha_{1,n}, \alpha_{2,n}, \dots, \alpha_{k,n}) \in \mathbb{R}^k$ where $n \in \mathbb{N}$, then
  - $\boldsymbol{\{\overrightarrow{x_n}\}}$ **converges to** $\boldsymbol{(\alpha_1, \alpha_2, \dots, \alpha_k) \Leftrightarrow \lim_{n\to\infty} \alpha_{j,n} = \alpha_j}$ $(1 \le j \le k)$
- Proof (a)
  - Assume $\overrightarrow{x_n} \to \vec{x}$
    - Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $|\overrightarrow{x_n} - \vec{x}| < \varepsilon$ for $n \ge N$
    - Thus, $|\alpha_{j,n} - \alpha_j| \le |\overrightarrow{x_n} - \vec{x}|$ for $n \ge N, 1 \le j \le k$
    - Therefore $\lim_{n\to\infty} \alpha_{j,n} = \alpha_j$ for $1 \le j \le k$
  - Assume $\lim_{n\to\infty} \alpha_{j,n} = \alpha_j$ for $1 \le j \le k$
    - Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $|\alpha_{j,n} - \alpha_j| < \frac{\varepsilon}{\sqrt{k}}$ for $n \ge N$
    - $|\overrightarrow{x_n} - \vec{x}| = \left|\sqrt{\sum_{i=1}^{k} |\alpha_{j,n} - \alpha_n|^2}\right| = \sqrt{\sum_{i=1}^{k} |\alpha_{j,n} - \alpha_n|^2} < \sqrt{\sum_{i=1}^{k} \frac{\varepsilon^2}{k}} = \varepsilon$
    - Therefore $\overrightarrow{x_n} \to \vec{x}$
- Statement (b)
  - Suppose
    - $\{\overrightarrow{x_n}\}$ and $\{\overrightarrow{y_n}\}$ are sequences in $\mathbb{R}^k$, $\{\beta_n\}$ is a sequence in $\mathbb{R}$
    - $\overrightarrow{x_n} \to \vec{x}, \overrightarrow{y_n} \to \vec{y}, \beta_n \to \beta$
  - Then
    - $\boldsymbol{\lim_{n\to\infty} \overrightarrow{x_n} + \overrightarrow{y_n} = \vec{x} + \vec{y}}$
    - $\boldsymbol{\lim_{n\to\infty} \overrightarrow{x_n} \cdot \overrightarrow{y_n} = \vec{x} \cdot \vec{y}}$
    - $\boldsymbol{\lim_{n\to\infty} \beta_n \cdot \overrightarrow{x_n} = \beta \cdot \vec{x}}$
- Proof (b)
  - This follows from (a) and Theorem 3.3 (Algebraic Limit Theorem)

Definition 2.31: Open Cover

- An **open cover** of a set E in a metric X is
- a collection of open sets $\{G_\alpha\}$ in X s.t. $E \subset \bigcup_\alpha G_\alpha$

Definition 2.32: Compact Sets

- Definition
    - A set K in a metric space X is **compact** if
    - **every open cover** of K has a **finite subcover**
- Intuition for $\mathbb{R}^k$: **Closed and bounded**



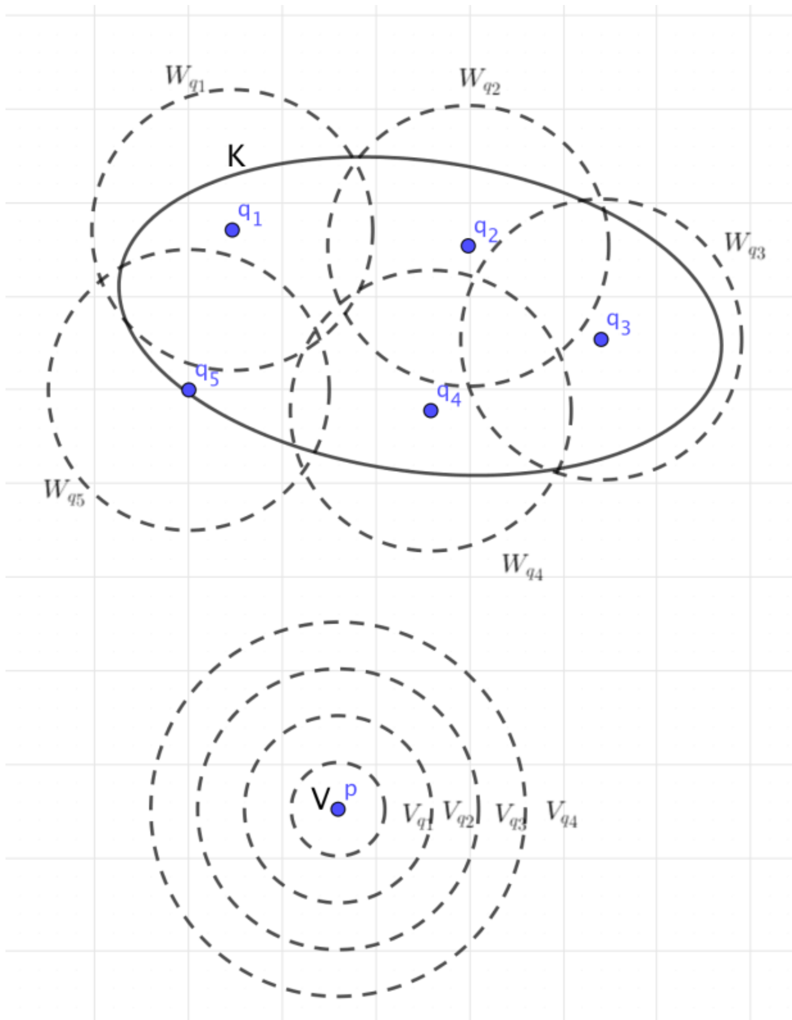
- Example 1
    - Let $E = (0,1), X = \mathbb{R}$
    - E is a open cover of itself, but E is not compact
    - Let $G_\alpha = \left(\frac{\alpha}{2}, 1\right)$ for $\alpha \in (0,1)$, then E has $\{G_n\}$ as an open cover
    - We cannot take a finite collection of these G_α and still have an open cover
    - So it has no finite subcover
    - Therefore $E = (0,1)$ is not compact
- Example 2
    - Let $K = [0,1], X = \mathbb{R}$
    - Consider $\{G_\alpha\} \cup \{G_0\} \cup \{G_1\}$, where
        - $G_\alpha = \left(\frac{\alpha}{2}, 1\right)$ for $\alpha \in (0,1)$
        - $G_0 = (-\varepsilon, \varepsilon)$
        - $G_1 = (1-\varepsilon, 1+\varepsilon)$ for some $\varepsilon > 0$
    - Then $\{G_\alpha\} \cup \{G_0\} \cup \{G_1\}$ is an open cover of $[0,1]$
    - It has finite subcover $\{G_0, G_1, G_\varepsilon\}$ where $G_\varepsilon = \left(\frac{\varepsilon}{2}, 1\right)$
    - Therefore $K = [0,1]$ is compact

Compact Subset, Cantor's Intersection Theorem

Monday, March 12, 2018 12:08 PM

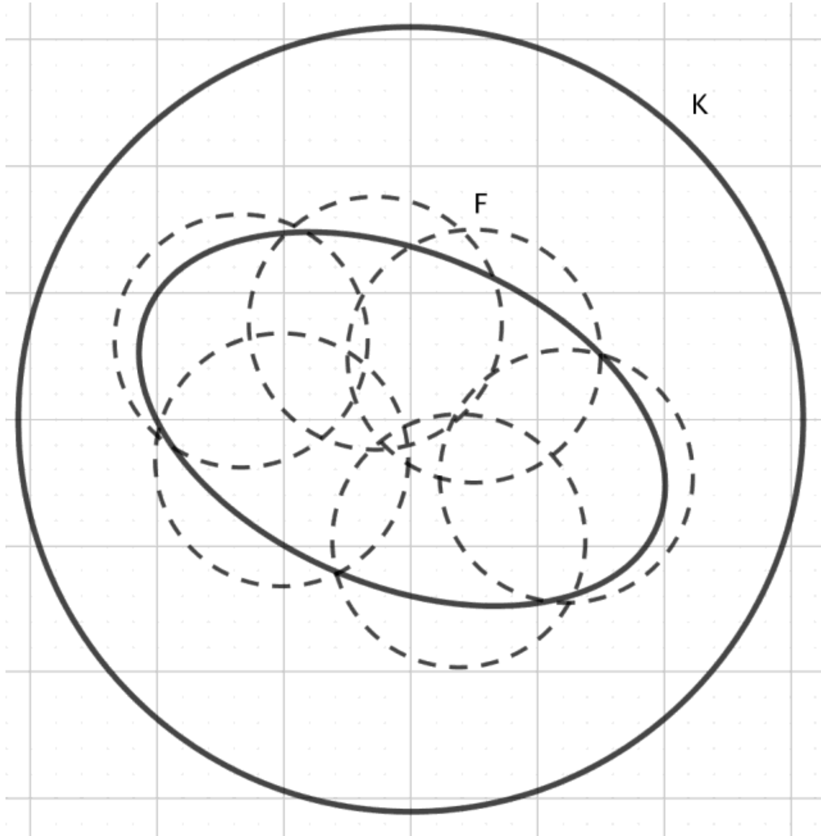
Theorem 2.34: Compact Sets are Closed

- Statement
  - **Compact** subsets of metric spaces are **closed**
- Proof
  - Let K be a compact subset of a metric space X
  - We shall prove that the complement of K is open
  - Let $p \in K^c, q \in K$
  - Let $V_q = N_r(p), W_q = N_s(q)$ where $r, s < \frac{1}{2}d(p, q)$
  - Since K is compact, $\exists q_1, q_2, \ldots q_n \in K$ s.t.
  - $K \subset W_{q_1} \cup W_{q_2} \cup \cdots \cup W_{q_n} = W$
  - Let $V = V_{q_1} \cap V_{q_2} \cap \cdots \cap V_{q_n}$
  - Then V is a neighborhood of p that does not intersect W
  - $V \subset K^c \Rightarrow p$ is an interior point of K^c
  - So K^c is open and therefore K is closed



Theorem 2.35: Closed Subsets of Compact Sets are Compact

- Statement
  - **Closed subsets** of **compact** sets are **compact**
- Proof
  - Let X be a metric space
  - Suppose $F \subset K \subset X$, where F is closed, and K is compact
  - Let $\{V_\alpha\}$ be an open cover of F
  - Consider $\{V_\alpha\} \cup \{F^c\}$, where F^c is open
  - Then $\{V_\alpha\} \cup \{F^c\}$ is an open cover of K
  - Since K is compact, K has a finite subcover Φ
  - If $F^c \in \Phi$, then $\Phi \setminus \{F^c\}$ is still finite and covers F
  - So we have a finite subcover of $\{V_\alpha\}$
  - Therefore F is compact



- Corollary
  - If F is closed and K is compact , then $F \cap K$ is compact
- Proof
  - K compact $\Rightarrow K$ is closed
  - We know F is closed, so $F \cap K$ is closed
  - $F \cap K \subset K$, and K is compact
  - So $F \cap K$ is compact

Theorem 2.36: Cantor's Intersection Theorem

- Statement
  - If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X s.t.
  - The intersection of every **finite subcollection** of $\{K_\alpha\}$ is **nonempty**
  - Then $\bigcap_\alpha K_\alpha$ is **nonempty**
- Proof
  - Fix $K_1 \in \{K_\alpha\}$ and let $G_\alpha = K_\alpha^c, \forall \alpha$
  - Assume no point of K_1 belongs to every K_α
  - Then $\{G_\alpha\}$ is an open cover of K_1
  - Since K_1 is compact, $K_1 \subset G_{\alpha_1} \cap G_{\alpha_2} \cap \cdots \cap G_{\alpha_n}$
  - Where $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a finite collection of indices
  - Then $K_1 \cap G_{\alpha_2} \cap \cdots \cap G_{\alpha_n} = \emptyset$

- This is a contradiction, so no such set K_1 exists
- The result follows

- Corollary
  - If $\{K_n\}$ is a sequence of nonempty compact sets s.t. $K_n \supset K_{n+1}, \forall n \in \mathbb{N}$
  - Then $\bigcap_{n=1}^{\infty} K_n$ is nonempty

Theorem 2.37: Infinite Subset of Compact Set

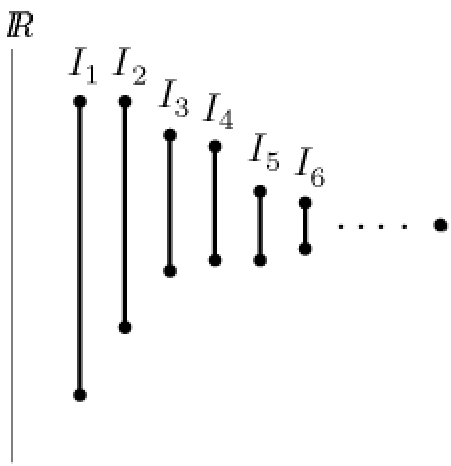
- Statement
  - If E is an **infinite subset** of a compact set K
  - Then E has a **limit point** in K
- Proof
  - If no point of K were a limit point of E
  - Then $\forall q \in K, \exists N(q)$ s.t. no point of E other than q
  - i.e. $N(q)$ contains at most one point of E (namely, q, if $q \in E$)
  - So no finite sub-collection of $\{N(q)\}$ can cover E, and thus not K
  - This is a contradiction, so E has a limit point in K

Nested Intervals Theorem, Compactness of k-cell

Wednesday, March 14, 2018 12:06 PM

Theorem 2.38: Nested Intervals Theorem

- Statement
  - If $\{I_n\}$ is a **sequence of closed intervals** in $\mathbb{R}$ s.t. $I_n \supset I_{n+1}, \forall n \in \mathbb{N}$
  - Then $\bigcap_{n=1}^{\infty} I_n$ is **nonempty**
- Intuition



- Proof
  - Let $I_n \coloneqq [a_n, b_n]$
  - Let $E \coloneqq \{a_n\}_{n\in\mathbb{N}}$
    - E is nonempty
    - E is bounded above by b_1 since $b_1 \geq a_n, \forall n \in \mathbb{N}$
    - So $\sup E$ exists
  - Let $x \coloneqq \sup E$
  - $\forall m, n \in \mathbb{N}, a_n \leq a_{m+n} \leq b_{m+n} \leq b_m$
    - $a_n \leq b_m \Rightarrow x \leq b_m, \forall m \in \mathbb{N}$
    - $x = \sup E \Rightarrow a_m \leq x, \forall m \in \mathbb{N}$
  - So, $x \in [a_m, b_m], \forall m \in \mathbb{N}$
  - Therefore $x \in \bigcap_{n=1}^{\infty} I_n$

Theorem 2.39: Nested k-cell

- Statement
  - Let k be a positive integer
  - If $\{I_n\}$ is a **sequence of k-cells** s.t. $I_n \supset I_{n+1}, \forall n \in \mathbb{N}$

- Then $\bigcap_{n=1}^{\infty} I_n$ is **nonempty**

- Proof
  - Let I_n consists of all points $\vec{x} = (x_1, x_2, \dots, x_k)$ s.t.
  - $a_{n,j} \le x_j \le b_{n,j}$, where $1 \le j \le k, n = 1,2,3, \dots$
  - Let $I_{n,j} = [a_{n,j}, b_{n,j}]$
  - For each j, $\{I_{n,j}\}$ satisfies the hypothesis of Theorem 2.38
  - Therefore $\exists x_j^* \in \bigcap_{n=1}^{\infty} I_{n,j}$, for $1 \le j \le k$
  - Let $\overrightarrow{x^*} = (x_1^*, x_2^*, \dots, x_k^*)$
  - By construction, $\overrightarrow{x^*} \in \bigcap_{n=1}^{\infty} I_n$

Theorem 2.40: Compactness of k-cell

- Statement
  - Every ***k*-cell** is **compact**
- Proof
  - Let $I = \left\{(x_1, x_2, \dots, x_k) \in \mathbb{R}^k \middle| a_j \le x_j \le b_j, 1 \le j \le k\right\}$ be a k-cell
  - Let $\delta = \sqrt{\sum_{j=1}^{k} (b_j - a_j)^2}$, then $|\vec{x} - \vec{y}| \le \delta, \forall \vec{x}, \vec{y} \in I$
  - Suppose $\{G_\alpha\}$ is an open cover of I with no finite subcover
  - Build sequence $\{I_n\}$
    - Let $c_j = \frac{a_j + b_j}{2}$
    - Consider intervals $[a_j, c_j]$ and $[c_j, b_j]$
    - Those intervals describes 2^k k-cells Q_i whose union is I
    - Since the number of Q_i is finite, and $\{G_\alpha\}$ has no finite subcover
    - $\exists Q_i$ not covered by a finite subcover of $\{G_\alpha\}$; call this I_1
    - Repeat this process on I_1 to obtain $I_2, I_3, \dots$
    - We can build a sequence $\{I_n\}$
  - $\{I_n\}$ is a sequence of k-cells s.t.
    - $I \supset I_1 \supset I_2 \supset \cdots$
    - I_n is not covered by any finite sub-collection of $\{G_\alpha\}$
    - If $\vec{x}, \vec{y} \in I_n$, then $|\vec{x} - \vec{y}| \le \frac{\delta}{2^n}$

- By Theorem 2.38, $\exists \vec{x}^* \in I_n, \forall n \in \mathbb{N}$
- Then $\overrightarrow{x^*} \in G_\alpha$, for some G_α
  - G_α is open
  - i.e. $\exists r > 0$ s.t. $\left|\vec{y} - \overrightarrow{x^*}\right| < r \Rightarrow \vec{y} \in G_\alpha$
  - By Archimedean Property, $\exists n \in \mathbb{N}$ s.t. $\frac{\delta}{2^n} < r$
  - In this case, $I_n \subset G_\alpha$, which is impossible, since
  - I_n is not covered by any finite sub-collection of $\{G_\alpha\}$
  - So no such open cover $\{G_\alpha\}$ exists
- So every open cover of I have a finite subcover
- Therefore I is compact

Heine-Borel, Weierstrass, Subsequence

Friday, March 16, 2018 12:07 PM

Theorem 2.41: The Heine-Borel Theorem

- For a set $E \subset \mathbb{R}^k$, the following properties are equivalent
  - (a) E is **closed** and **bounded**
  - (b) E is **compact**
  - (c) Every **infinite subset** of E has a **limit point** in E
- Proof $(a) \Rightarrow (b)$
  - If (a) holds, then $E \subset I$ for some k-cell
  - (b) follow from
    - Theorem 2.40 (I is compact)
    - Theorem 2.35 (Closed subsets of compact sets are compact)
- Proof $(b) \Rightarrow (c)$
  - See Theorem 2.37
- Proof $(c) \Rightarrow (a)$
  - Suppose E is not bounded
    - $\exists x_n \in E$ s.t. $|\overrightarrow{x_n}| > n, \forall n \in \mathbb{N}$
    - $\{\overrightarrow{x_n}\}$ is an infinite subset of E with no limit points
    - This is a contradiction, so E must be bounded
  - Suppose E is not closed
    - $\exists \overrightarrow{x_0} \in \mathbb{R}^k$ that is a limit point of E but not in E
    - For $n \in \mathbb{N}, \exists \overrightarrow{x_n} \in E$ s.t. $|\overrightarrow{x_n} - \overrightarrow{x_0}| < \frac{1}{n}$
    - Let $S \coloneqq \{\overrightarrow{x_n}\}_{n \in \mathbb{N}}$ be a infinite subset of E
    - By construction, S has $\overrightarrow{x_0}$ as a limit point
    - We want to show that $\overrightarrow{x_0}$ is the only limit point of S
      - Let $\vec{y} \in \mathbb{R}^k$ and $\vec{y} \neq \overrightarrow{x_0}$
      - By triangle inequality,
      - $|\overrightarrow{x_n} - \vec{y}| \geq |\overrightarrow{x_0} - \vec{y}| - |\overrightarrow{x_n} - \overrightarrow{x_0}| \geq |\overrightarrow{x_0} - \vec{y}| - \frac{1}{n} > \frac{1}{2}|\overrightarrow{x_0} - \vec{y}|$
      - For all but finitely many n
      - Take the neighborhood of $\vec{y}$ with radius of $\frac{1}{2}|\overrightarrow{x_0} - \vec{y}|$,
      - There are only finitely many points of S in it
      - By Theorem 2.20, $\vec{y}$ cannot be a limit point of S
      - Since $\vec{y}$ was arbitrary, $\overrightarrow{x_0}$ is the only limit point of S

- By (c), S has a limit point in E i.e. $\overrightarrow{x_0} \in E$
- This is a contradiction, so E has to be closed

- Therefore E is closed and bounded

Theorem 2.42: The Weierstrass Theorem

- Statement
  - Every **bounded infinite subset** E of $\mathbb{R}^k$ has a **limit point** in $\mathbb{R}^k$
- Proof
  - E is bounded, so $E \subset I \subset \mathbb{R}^k$ for some k-cell I
  - By Theorem 2.40, I is compact
  - By Theorem 2.37, E has a limit point in I
  - Hence, E has a limit point in $\mathbb{R}^k$

Definition 3.5: Subsequences

- Definition
  - Given a sequence $\{p_n\}$
  - Consider a sequence $\{n_k\} \subset \mathbb{N}$ with $n_1 < n_2 < n_3 < \cdots$
  - Then the sequence $\{p_{n_i}\}$ is a **subsequence** of $\{p_n\}$
  - If $\{p_{n_i}\}$ converges, its limit is called a **subsequential limit** of $\{p_n\}$
- Example
  - Let $\{p_n\} = \frac{1}{n} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right\}$
  - One subsequence is $\left\{1, \frac{1}{4}, \frac{1}{6}, \frac{1}{7}, \frac{1}{38}, \frac{1}{101}, \frac{1}{135}, \dots\right\}$
  - But $\left\{\frac{1}{19}, \frac{1}{18}, \frac{1}{2}, \frac{1}{237}, \frac{1}{12}, \frac{1}{59}, \frac{1}{32}, \dots\right\}$ is not a subsequence
- Note
  - A subsequential limit might exist for a sequence in the absence of a limit
  - $\{p_n\}$ converges to p if and only if every subsequence of $\{p_n\}$ converges to p

Theorem 3.6: Properties of Subsequence

- Statement (a)
  - If $\{p_n\}$ is a sequence in a compact metric space X
  - Then **some subsequence** of $\{p_n\}$ **converges** to a point of X
- Proof (a)
  - Let E be the range of $\{p_n\}$
  - If E is finite
    - $\exists p \in E$ and a sequence $\{n_i\} \subset \mathbb{N}$ with $n_1 < n_2 < n_3 < \cdots$ s.t.

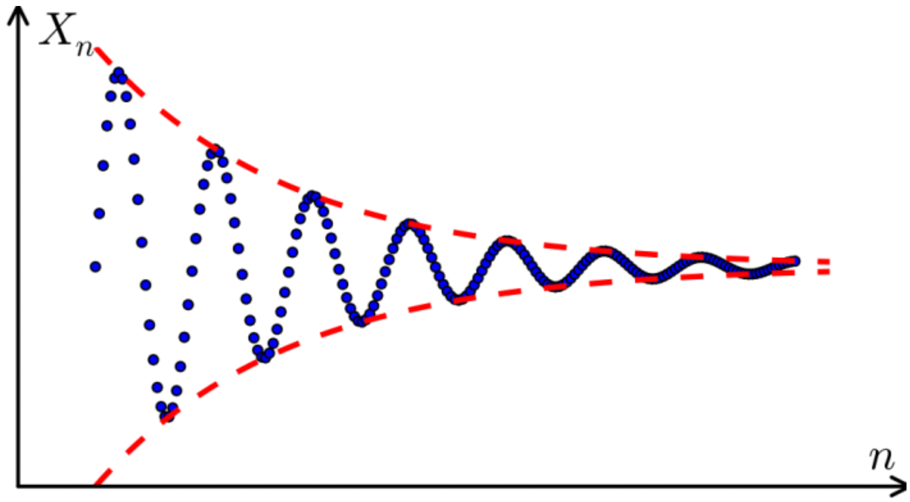
            - $p_{n_1} = p_{n_2} = p_{n_3} = \cdots = p$
    - If E is infinite
        - By Theorem 2.37, E has a limit point $p \in X$
        - By Theorem 2.20, inductively choose n_i s.t. $d\left(p, p_{n_i}\right) < \frac{1}{i}, \forall i \in \mathbb{N}$
        - It follows that $\left\{p_{n_i}\right\}$ converges to p
- Statement (b)
    - Every **bounded sequences** in $\mathbb{R}^k$ contains a **convergent subsequence**
- Proof (b)
    - By Theorem 2.41, every bounded subset of $\mathbb{R}^k$ is in a compact subset of $\mathbb{R}^k$
    - Result follows by (a)

Cauchy Sequence, Diameter

Monday, March 19, 2018 12:19 PM

Definition 3.8: Cauchy Sequence

- A sequence $\{p_n\}$ in a metric space X is said to be **Cauchy sequence**
- If $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\boldsymbol{d(p_n, p_m) < \varepsilon}, \forall n, m \geq N$



Definition 3.9: Diameter

- Let E be a nonempty subset of metric space X
- Let S be set of all real numbers of the form $d(p, q)$ with $p, q \in E$
- Then diam $S \coloneqq \sup S$ is called the **diameter** of E (possibly ∞)
- If $\{p_n\}$ is a sequence in X and $E = \{p_N, p_{N+1}, \dots\}$
- Then $\{p_n\}$ is a **Cauchy sequence** if and only if $\boldsymbol{\lim_{N \to \infty} \operatorname{diam} E_N = 0}$

Theorem 3.10: Diameter and Closure

- Statement
  - If $\bar{E}$ is the closure of a set E in a metric space X, then $\boldsymbol{\operatorname{diam} \bar{E} = \operatorname{diam} E}$
- Proof
  - diam $E \leq$ diam $\bar{E}$
    - This is obvious since $E \subset \bar{E}$
  - diam $\bar{E} \leq$ diam E
    - Let $p, q \in \bar{E}$
    - Let $\varepsilon > 0$, then $\exists p', q' \in E$ s.t. $d(p, p') < \frac{\varepsilon}{2}, d(q, q') < \frac{\varepsilon}{2}$
    - $d(p, q) \leq$ diam E
      - $d(p, q) \leq d(p, p') + d(p', q') + d(q', q)$
      - $< \frac{\varepsilon}{2} + d(p', q') + \frac{\varepsilon}{2}$

- $= \varepsilon + d(p', q')$
- $\leq \varepsilon + \text{diam } E$
- Since $\varepsilon > 0$ was arbitrary, $d(p, q) \leq \text{diam } E$

  - So $\text{diam } \bar{E} \leq \text{diam } E$

- Therefore $\text{diam } \bar{E} = \text{diam } E$

Theorem 3.10: Nested Compact Set

- Statement
  - If K_n is a sequence of compact sets in X s.t.
  - $\boldsymbol{K_n \supset K_{n+1}}, \forall n$ and $\boldsymbol{\lim_{n \to \infty} \text{diam } K_n = 0}$
  - Then $\bigcap_{n=1}^{\infty} K_n$ consists of **exactly one point**
- Proof
  - Let $K = \bigcap_{n=1}^{\infty} K_n$
  - By Theorem 2.36, K is not empty
  - If K contains more than one point, $\text{diam } K > 0$
  - But $K_n \supset K, \forall n \in \mathbb{N}$, then
  - $\text{diam } K_n \geq \text{diam } K > 0 \Rightarrow \lim_{n \to \infty} K_n \geq \text{diam } K > 0$
  - This contradicts $\lim_{n \to \infty} \text{diam } K_n = 0$
  - There can only be one point in K

Cauchy Sequence, Complete Metric Space, Monotonic

Wednesday, March 21, 2018 12:07 PM

Theorem 3.11: Cauchy Sequence and Convergence

- Statement (a)
  - In any metric space X, every **convergent** sequence is a **Cauchy sequence**
- Proof (a)
  - Suppose $p_n \to p$
  - Let $\varepsilon > 0$, then $\exists N \in \mathbb{N}$ s.t. $d(p, p_n) < \frac{\varepsilon}{2}, \forall n \geq N$
  - $d(p_n, p_m) \leq d(p, p_n) + d(p, p_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \forall n, m \geq N$
  - So $\{p_n\}$ is a Cauchy sequence
- Statement (b)
  - If X is a **compact** metric space and $\{p_n\}$ is a **Cauchy sequence**
  - Then $\{p_n\}$ **converges** to some point of X
- Proof (b)
  - Let $\{p_n\}$ be a Cauchy sequenece in compact metric space X
  - For $N \in \mathbb{N}$, let $E_N = \{p_N, p_{N+1}, \dots\}$
  - By Theorem 3.10, $\lim_{N\to\infty} \operatorname{diam} \overline{E_N} = \lim_{N\to\infty} \operatorname{diam} E_N = 0$
  - By Theorem 2.35, $\overline{E_N}$ as closed subset of X is compact
  - Since $E_{N+1} \subset E_N, \overline{E_{N+1}} \subset \overline{E_N}, \forall N \in \mathbb{N}$
  - By Theorem 3.10 (b), $\exists!\, p \in X$ s.t. $p \in \overline{E_N}, \forall N \in \mathbb{N}$
  - Let $\varepsilon > 0$ be given, $\exists N_0 \in \mathbb{N}$ s.t. diam $\overline{E_N} < \varepsilon, \forall N \geq N_0$
  - Since $p \in \overline{E_N}, d(p, q) < \varepsilon, \forall q \in E_N = \{p_N, p_{N+1}, \dots\} \subset \overline{E_N}$
  - In other word, $d(p, p_n) < \varepsilon$ for $n \geq N_0$
  - So $\lim_{n\to\infty} p_n = p$
- Statement (c)
  - In $\mathbb{R}^k$, every **Cauchy sequence converges**
- Proof (c)
  - Let $\{\overrightarrow{x_n}\}$ be a Cauchy sequence in $\mathbb{R}^k$
  - Let $E_N = \{\overrightarrow{x_N}, \overrightarrow{x_{N+1}}, \dots\}$
  - For some $N \in \mathbb{N}$, diam $E_N < 1$
  - Then the range of $\{\overrightarrow{x_n}\}$ is $\{\overrightarrow{x_1}, \dots, \overrightarrow{x_{N-1}}\} \cup E_N$
  - By Theorem 2.41, every bounded subset of $\mathbb{R}^k$ has compact closure in $\mathbb{R}^k$

- (c) follows from (b)

Definition 3.12: Complete Metric Space

- Definition
  - A metric space X is said to be complete if
  - **every Cauchy sequence converges** in X
- Examples
  - $\mathbb{R}^k$ is complete
  - Compact metric space X is complete
  - $\mathbb{Q}$ is not complete (convergence may lie outside of $\mathbb{Q}$)

Definition 3.13: Monotonic Sequence

- A sequence $\{s_n\}$ of real numbers is said to be
- **monotonically increasing** if $s_n \leq s_{n+1}, \forall n \in \mathbb{N}$
- **monotonically decreasing** if $s_n \geq s_{n+1}, \forall n \in \mathbb{N}$
- **monotonic** if $\{s_n\}$ is either monotonically increasing or decreasing

Theorem 3.14: Monotone Convergence Theorem

- Statement
  - If $\{s_n\}$ is **monotonic**, then $\{s_n\}$ **converges** if and only if it is **bounded**
- Proof
  - By Theorem 3.2 (c), converge implies boundedness
  - Without loss of generality, suppose $\{s_n\}$ is monotonically increasing
  - Let $E = \text{range}\,\{s_n\}$, and $s = \sup E$, then $s_n \leq s, \forall n \in \mathbb{N}$
  - Given $\varepsilon > 0, \exists N \in \mathbb{N}\ \text{s.t.}\ s - \varepsilon < s_n \leq s, \forall n \geq N$
  - Since $s - \varepsilon$ is not an upper bound of E, and $\{s_n\}$ is increasing
  - $s - s_n < \varepsilon, \forall n \geq N \Rightarrow \lim\limits_{n\to\infty} s_n = s$

Upper and Lower Limits

Friday, March 23, 2018 12:11 PM

Definition 3.15: Sequences Approaching Infinity

- Let $\{s_n\}$ be a sequence of real numbers s.t.
- $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. $s_n \geq M, \forall n \geq N$
- Then we write $s_n \to +\infty$
- Similarly if $\forall M \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. $s_n \leq M, \forall n \geq N$
- Then we write $s_n \to -\infty$

Definition 3.16: Upper and Lower Limits

- Definition
  - Let $\{s_n\}$ be a sequence of real numbers
  - Let E be the set of x (in the extended real number system) s.t.
  - $s_{n_k} \to x$ for some subsequence $\{s_{n_k}\}$
  - E contains **all subsequential limits** of $\{s_n\}$ plus possibly $+\infty, -\infty$
  - $\limsup_{n\to\infty} s_n = s^* = \sup E$ is called the **upper limit** of $\{s_n\}$
  - $\liminf_{n\to\infty} s_n = s_* = \inf E$ is called the **lower limit** of $\{s_n\}$
- Example 1
  - $s_n = \frac{(-1)^n}{1+\frac{1}{n}} = \left\{-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \dots\right\}$
  - $\limsup_{n\to\infty} s_n = \sup\{-1,1\} = 1$
  - $\liminf_{n\to\infty} s_n = \inf\{-1,1\} = -1$
- Example 2
  - $\lim_{n\to\infty} s_n = s \Rightarrow \limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s$
    - All subsequential limits of a convergent sequence
    - converge to the same value as the sequence
  - $\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s \Rightarrow \lim_{n\to\infty} s_n = s$
    - $\Rightarrow \sup E = \inf E$
    - $\Rightarrow E = \{s\}$
    - $\Rightarrow$ All subsequential limits $= s$
    - $\Rightarrow \lim_{n\to\infty} s_n = s$

Theorem 3.17: Properties of Upper Limits

- Let $\{s_n\}$ be a sequence of real numbers, then
- $\boldsymbol{s^* \in E}$
  - When $s^* = +\infty$
    - E is not bounded above, so $\{s_n\}$ is not bounded above
    - There is a subseqnence $\{s_{n_k}\}$ s.t. $s_{n_k} \to \infty$
    - So $s^* = +\infty \in E$
  - When $s^* \in \mathbb{R}$
    - E is bounded above
    - And at least one subsequential limit exists i.e. $E \neq \emptyset$
    - By Theorem 3.7, E is closed i.e. $E = \bar{E}$
    - By Theorem 2.28, $s^* = \sup E \in \bar{E}$
    - Therefore $s^* \in E$
  - When $s^* = -\infty$
    - Then $E = \{-\infty\}$
    - $s_n \to -\infty$ and $s^* = -\infty \in E$
- **If $\boldsymbol{x > s^*}$, then $\boldsymbol{\exists N \in \mathbb{N}}$ s.t. $\boldsymbol{s_n < x}$ for $\boldsymbol{n \geq N}$**
  - If $\exists x > s^*$ with $s_n \geq x$ for infinitely many $n \in \mathbb{N}$
  - Then $\exists y \in E$ s.t. $y \geq x > s^*$
  - This contradicts the definition of s^*
- Moreover s^* is the **only number** with these properties
  - Suppose $p, q \in E, p \neq q$ s.t. the property above holds for p, q
  - Without loss of generality, suppose $p < q$
  - Choose x s.t. $p < x < q$
  - Since p satisfies the property above
  - $\exists N \in \mathbb{N}$ s.t. $s_n < x, \forall n \geq N$
  - So no subsequence of $\{s_n\}$ can converge to q
  - This contradicts the existence of q
  - Therefore only one number can have these properties

Some Special Sequences

Monday, April 2, 2018 12:11 PM

Theorem 3.20: Some Special Sequences

- Lemma (The Squeeze Theorem)
  - Given $0 \le x_n \le s_n$, for $n \ge N$ where $N \in \mathbb{N}$ is some fixed number
  - If $s_n \to 0$, then $x_n \to 0$
  - (Proof on homework)
- **If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$**
  - For $n \ge N$, we need $\left|\frac{1}{n^p} - 0\right| < \varepsilon \Rightarrow n > \frac{1}{\varepsilon^{1/p}}$
  - Given $\varepsilon > 0$
  - Using Archimedean Property, take $N > \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}$
  - So, for $n \ge N$, $n > \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}} \Rightarrow n^p > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n^p} < \varepsilon \Rightarrow \left|\frac{1}{n^p} - 0\right| < \varepsilon$
  - Therefore $\lim_{n\to\infty} \frac{1}{n^p} = 0$
- **If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$**
  - When $p = 1$
    - We are done, since $\lim_{n\to\infty} 1 = 1$
  - When $p > 1$
    - Then $p - 1 > 0$
    - Let $x_n = \sqrt[n]{p} - 1$, then $x_n > 0$
    - $p = (x_n + 1)^n \ge 1^n + \binom{n}{n-1} 1^{n-1} x_n^1 = 1 + n x_n$
    - $\Rightarrow p - 1 \ge n x_n$
    - $\Rightarrow \frac{p-1}{n} \ge x_n > 0$
    - By the Squeeze Theorem, $x_n \to 0$
    - i.e. $\lim_{n\to\infty} \sqrt[n]{p} - 1 = 0$
    - So $\lim_{n\to\infty} \sqrt[n]{p} = 1$
  - When $p < 1$
    - Then $\frac{1}{p} > 1$
    - So, $\lim_{n\to\infty} \sqrt[n]{1/p} = 1$

- Therefore $\lim_{n\to\infty} \sqrt[n]{p} = \frac{1}{1} = 1$

- $\boldsymbol{\lim_{n\to\infty} \sqrt[n]{n} = 1}$
  - Let $x_n = \sqrt[n]{n} - 1 \geq 0$
  - $n = (x_n + 1)^n \geq \binom{n}{n-2} 1^{n-2} x_n^2 = \frac{n!}{(n-2)!\,2!} x_n^2 = \frac{n(n-1)}{2} x_n^2$
  - $\Rightarrow \frac{2}{n-1} \geq x_n^2$
  - $\Rightarrow \sqrt{\frac{2}{n-1}} \geq x_n > 0$ for $n > 1$
  - By the Squeeze Theorem, $x_n = \lim_{n\to\infty} \sqrt[n]{n} - 1 \to 0$
  - i.e. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

- **If $\boldsymbol{p > 0, \alpha \in \mathbb{R}}$, then $\boldsymbol{\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0}$**
  - Let $k \in \mathbb{N}$ s.t. $k > \alpha$ by Archimedean Property
  - For $n > 2k$, $(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\cdots(n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!}$
  - Because $n > 2k \Rightarrow \frac{n}{2} > k \Rightarrow n - k > \frac{n}{2} \Rightarrow n - k + 1 > \frac{n}{2}$
  - So, $0 < \frac{n^\alpha}{(1+p)^\alpha} < \frac{2^k k!}{n^k p^k} \cdot n^\alpha = \frac{2^k k!}{p^k} \cdot n^{\alpha - k}$
  - Since $\alpha - k < 0$, $n^{\alpha-k} \to 0 \Rightarrow \frac{2^k k!}{p^k} \cdot n^{\alpha-k} \to 0$
  - By the Squeeze Theorem, $\frac{n^\alpha}{(1+p)^\alpha} \to 0$
  - i.e. $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$

- **If $\boldsymbol{|x| < 1}$, then $\boldsymbol{\lim_{n\to\infty} x^n = 0}$**
  - $|x| < 1 \Rightarrow \frac{1}{|x|} > 1$
  - Let $p = \frac{1}{|x|} - 1 > 0$
  - Take $\alpha = 0$ in the limit above, we get $\lim_{n\to\infty} \frac{1}{(1+p)^n} = 0$
  - So $\lim_{n\to\infty} \frac{1}{\left(1 + \frac{1}{|x|} - 1\right)^n} = \lim_{n\to\infty} |x|^n = 0$
  - Then $\lim_{n\to\infty} x^n = 0$

Series, Cauchy Criterion for Series, Comparison Test

Wednesday, April 4, 2018 12:09 PM

Definition 3.31: Series

- Given a sequence $\{a_n\}$
- We associate a **sequence of partial sums** $\{s_n\}$ where
- $s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$
- $\sum_{k=1}^{\infty} a_k$ is called an **infinite series**, or simply series
- If $\{s_n\}$ diverges, the series is said to diverge
- If $\{s_n\}$ converges to s, the series is said to converge, and write $\sum_{k=1}^{\infty} a_k = s$
- s is called the **sum of the series**
- But it is technically the limit of a sequence of sums

Theorem 3.22: Cauchy Criterion for Series

- Statement
    - $\sum_{n=1}^{\infty} a_n$ **converges** $\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\left|\boldsymbol{\sum_{k=n}^{m} a_k}\right| < \boldsymbol{\varepsilon}, \forall m \geq n \geq N$
- Proof
    - This is Theorem 3.11 applied to $\{s_n\}$

Theorem 3.23: Series and Limit of Sequence

- Statement
    - In the setting of Theorem 3.22, take $m = n$
    - We have $|a_n| < \varepsilon$ for $n \geq N$
    - If $\boldsymbol{\sum_{n=1}^{\infty} a_n}$ **converges**, then $\boldsymbol{\lim_{n\to\infty} a_n = 0}$
- Note
    - If $a_n \to 0$, the series $\sum_{n=1}^{\infty} a_n$ might not converge
- Example: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges
    - $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \cdots$

- Therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Theorem 3.24: Convergence of Monotone Series

- Statement
  - A series of **nonnegative** real numbers **converges** if and only if
  - its **partial sum form a bounded sequence**
- Proof
  - See Theorem 3.14 (Monotone Convergence Theorem)

Theorem 3.25: Comparison Test

- If $|\boldsymbol{a_n}| < \boldsymbol{c_n}$ for $n \geq N_0 \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \boldsymbol{c_n}$ **converges**, then $\sum_{n=1}^{\infty} \boldsymbol{a_n}$ **converges**
  - Given $\varepsilon > 0, \exists N \geq N_0$ s.t. $\left|\sum_{k=n}^{m} c_k\right| = \sum_{k=n}^{m} c_k < \varepsilon$ for $m \geq n \geq N$
  - By the Cauchy Criterion, $\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k < \varepsilon$
  - Thus $\sum_{n=1}^{\infty} a_n$ converges
- If $\boldsymbol{a_n} \geq \boldsymbol{d_n} \geq \boldsymbol{0}$ for $n \geq N_0 \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \boldsymbol{d_n}$ **diverges**, then $\sum_{n=1}^{\infty} \boldsymbol{a_n}$ **diverges**
  - If $\sum_{n=1}^{\infty} a_n$ converges, then so must $\sum_{n=1}^{\infty} d_n$
  - This is a contradiction, so $\sum_{n=1}^{\infty} a_n$ diverges

Theorem 3.26: Convergence of Geometric Series

- Statement
  - If $\boldsymbol{0 < x < 1}$, then $\sum_{n=0}^{\infty} \boldsymbol{x^n} = \frac{\boldsymbol{1}}{\boldsymbol{1 - x}}$
  - If $\boldsymbol{x > 1}$, the series **diverges**
- Note
  - $\begin{cases} S = 1 + x + x^2 + \cdots \\ xS = x + x^2 + \cdots \end{cases} \Rightarrow S - xS = 1 \Rightarrow S = \frac{1}{1-x}$
  - This only works if we know this series converges
- Proof
  - If $0 < x < 1$, we have

- $\begin{cases} s_n = 1 + x + x^2 + \cdots + x^n \\ xs_n = x + x^2 + \cdots + x^n + x^{n+1} \end{cases}$
- $\Rightarrow s_n - xs_n = 1 - x^{n+1}$
- $\Rightarrow s_n = \dfrac{1 - x^{n+1}}{1 - x}$
- Since $0 < x < 1$, $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \dfrac{1 - x^{n+1}}{1 - x} = \dfrac{1}{1 - x}$
- Note if $x = 1$, $\sum\limits_{n=1}^{\infty} x^n = 1 + 1 + \cdots$ which diverges

Convergence Tests for Series

Friday, April 6, 2018 12:06 PM

Theorem 3.27: Cauchy Condensation Test

- Statement
  - Suppose $a_1 \geq a_2 \geq \cdots \geq 0$, then
  - $\sum_{n=1}^{\infty} \boldsymbol{a_n}$ **converges** $\Leftrightarrow \sum_{k=0}^{\infty} \boldsymbol{2^k a_{2^k}} = a_1 + 2a_2 + 4a_4 + \cdots$ **converges**
- Proof
  - By Theorem 3.24, we just need to look at boundness of partial sums
  - Let
    - $s_n = a_1 + a_2 + \cdots + a_n,$
    - $t_k = a_1 + 2a_2 + \cdots + 2^k a_{2^k}$
  - For $n \leq 2^k$
    - $s_n \leq a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1})$
    - $\leq a_1 + 2a_2 + \cdots + 2^k a_{2^k} = t^k$
  - For $n \geq 2^k$
    - $s_n \geq a_1 + (a_2 + a_3) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k})$
    - $\geq \frac{1}{2} a_1 + a_2 + \cdots + 2^{k-1} a_{2^k} = \frac{1}{2} t^k$
  - For $n = 2^k$
    - $s_n \leq t_k \leq 2s_n \Rightarrow s_{2^k} \leq t_k \leq 2s_{2^k}$
    - So $\{s_n\}$ and $\{t_k\}$ are both bounded or unbounded

Theorem 3.28: Convergence of p-Series

- Statement
  - $\sum_{n=1}^{\infty} \frac{\boldsymbol{1}}{\boldsymbol{n^p}}$ **converges if $\boldsymbol{p > 1}$ and diverges if $\boldsymbol{p \leq 1}$**
- Proof
  - If $p \leq 0$
    - Theorem 3.23 says if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$
    - In this case $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$, so series diverges
  - If $p > 0$
    - $\frac{1}{n^p} \geq \frac{1}{(n+1)^p}$ and $\frac{1}{n^p} \geq 0$

- By Cauchy Condensation Test,
- $\lim_{n\to\infty} \frac{1}{n^p}$ converges $\Leftrightarrow \sum_{n=1}^{\infty} 2^k \frac{1}{(2^k)^p}$ converges
- $\sum_{n=1}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{n=1}^{\infty} (2^{1-p})^k$ which is a geometric series
- By Theorem 3.26, this converges if $2^{1-p} < 1 \Leftrightarrow p > 1$
- Otherwise, $2^{1-p} > 1$, and this diverges

Theorem 3.33: Root Test

- Given $\sum_{n=1}^{\infty} a_n$, put $\boldsymbol{\alpha} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$, then
- **If $\alpha < 1$, $\sum_{n=1}^{\infty} a_n$ converges**
  - Theorem 3.17(b) says if $x > s^*$, then $\exists N \in \mathbb{N}$ s.t. $s_n < x$ for $n \geq N$
  - So let $\beta \in (\alpha, 1)$ and $N \in \mathbb{N}$ s.t. $\forall n \geq N$, $\sqrt[n]{|a_n|} < \beta$ i.e. $|a_n| < \beta^n$
  - $0 < \beta < 1$, so $\sum_{n=1}^{\infty} \beta^n$ converges
  - Thus, $\sum_{n=1}^{\infty} a_n$ converges by comparison test
- **If $\alpha > 1$, $\sum_{n=1}^{\infty} a_n$ diverges**
  - By Theorem 3.17, there exists a sequence $\{n_k\}$ s.t. $\sqrt[n_k]{|a_{n_k}|} \to \alpha$
  - So $|a_n| > 1$ for infinitely many n, i.e. $a_n \nrightarrow 0$
  - By Theorem 3.23, $\sum_{n=1}^{\infty} a_n$ diverges
- If $\alpha = 1$, this test gives no information
  - For $\sum_{n=1}^{\infty} \frac{1}{n}$, $\limsup_{n\to\infty} \sqrt[n]{n^{-1}} = \lim_{n\to\infty} \sqrt[n]{n^{-1}} = 1$, but the series diverges
  - For $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\limsup_{n\to\infty} \sqrt[n]{n^{-2}} = \lim_{n\to\infty} \frac{1}{(\sqrt[n]{n})^2} = 1$, but the series converges

Theorem 3.34: Ratio Test

- Statement
  - $\sum_{n=1}^{\infty} a_n$ **converges** if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

- $\displaystyle \boldsymbol{\sum_{n=1}^{\infty} a_n}$ **diverges** if $\left|\boldsymbol{\frac{a_{n+1}}{a_n}}\right| \geq \mathbf{1}, \forall n \geq n_0$ for some fixed $n_0 \in \mathbb{N}$

- Proof
  - If $\displaystyle \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$
  - We can find $\beta > 1, N \in \mathbb{N}$ s.t. $\left|\frac{a_{n+1}}{a_n}\right| < \beta, \forall n \geq N$
  - In particular
    - $|a_{N+1}| < \beta|a_N|$
    - $|a_{N+2}| < \beta|a_{N+1}| < \beta^2|a_N|$
    - $\vdots$
    - $|a_{N+p}| < \beta^p|a_N|$
  - So, $|a_n| < |a_N|\beta^{-N}\beta^n, \forall n \geq N$
  - $\beta < 1$, so $\displaystyle \sum_{n=1}^{\infty} \beta^n$ converges
  - So $\displaystyle \sum_{n=1}^{\infty} \underbrace{|a_N|\beta^{-N}}_{\text{constant}} \beta^n$ also converges
  - Therefore $\displaystyle \sum_{n=1}^{\infty} a_n$ converges by comparison test
  - On the other hand, if $|a_{n+1}| \geq |a_n|, \forall n \geq n_0 \in \mathbb{N}$
  - Then $a_n \nrightarrow 0$, so series divreges by Theorem 3.23

- Note
  - For $\displaystyle \sum_{n=1}^{\infty} \frac{1}{n}, \lim_{n\to\infty} \frac{1/n}{1/(n+1)} = 1$
  - For $\displaystyle \sum_{n=1}^{\infty} \frac{1}{n^2}, \lim_{n\to\infty} \frac{1/n^2}{1/(n+1)^2} = 1$
  - So $\displaystyle \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$ is not enough to conclude anything

Power Series, Absolute Convergence, Rearrangement

Monday, April 9, 2018 12:10 PM

Definition 3.38: Power Series

- Given a sequence $\{c_n\}$ of complex numbers
- The series $\boldsymbol{\sum_{n=1}^{\infty} c_n z^n}$ is a **power series**

Theorem 3.39: Convergence of Power Series

- Statement
  - Given the power sires $\sum_{n=1}^{\infty} c_n z^n$
  - Put $\alpha \coloneqq \limsup_{n\to\infty} \sqrt[n]{|c_n|}$
  - Let $R \coloneqq \frac{1}{\alpha}$ (If $\alpha = +\infty, R = 0$; If $\alpha = 0, R = +\infty$)
  - Then $\sum_{n=1}^{\infty} c_n z^n$ converges if $|z| < R$ and diverges if $|z| > R$
- Proof
  - Let $a_n = c_n z^n$ and apply the root test
  - $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \frac{|z|}{R}$
- Note: R is called the **radius of convergence** of the power series
- Examples
  - $\sum_{n=1}^{\infty} n^n z^n$ has $R = 0$
  - $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ has $R = +\infty$
  - $\sum_{n=0}^{\infty} z^n$ has $R = 1$. If $|z| = 1$, then the series diverges
  - $\sum_{n=1}^{\infty} \frac{z^n}{n}$ has $R = 1$, diverges if $z = 1$, converges for all other z with $|z| = 1$
  - $\sum_{n=1}^{\infty} \frac{z^n}{n^z}$ has $R = 1$, but converges for all z with $|z| = 1$ by comparison

Theorem 3.43: Alternating Series Test

- Statement

- Suppose we have a real sequence $\{c_n\}$ s.t.
  - $|c_1| \geq |c_2| \geq |c_3| \geq \cdots$
  - $c_{2m-1} \geq 0, c_{2m} \leq 0, \forall m \in \mathbb{N}$
  - $\lim_{n\to\infty} c_n = 0$
- Then $\sum_{n=1}^{\infty} c_n$ converges

- Proof: HW
- Example: alternating harmonic series
  - $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots$ converges to $\ln 2$

Absolute Convergence

- The series Σa_n is said to **converge absolutely** if the series $\Sigma|a_n|$ converges
- If Σa_n converges but $\Sigma|a_n|$ diverges
- We way that Σa_n **converges nonabsolutely or conditionally**

Theorem 3.45: Property of Absolute Convergence

- Statement
  - If Σa_n **converges absolutely**, then Σa_n **converges**
- Proof
  - $\left|\sum_{k=1}^{\infty} a_k\right| \leq \sum_{n=k}^{\infty} |a_k|$
  - The result follows by Cauchy Criterion

Definition 3.52: Rearrangement

- Let $\{k_n\}$ be a sequence in which **every natural number appears exactly once**
- Let $a'_n = a_{k_n}$, then $\Sigma a'_n$ is called a **rearrangement** of Σa_n

Theorem 3.54: Riemann Series Theorem

- Let Σa_n be a series of real number which **converges nonabsolutely**
- Let $-\infty \leq \alpha \leq \beta \leq +\infty$
- Then there **exists a rearrangement** $\Sigma a'_n$ s.t.
- $\boldsymbol{\liminf_{n\to\infty} s'_n = \alpha, \limsup_{n\to\infty} s'_n = \beta}$

Theorem 3.55: Rearrangement and Absolute Convergence

- Statement
  - If Σa_n is a series of complex numbers which **converges absolutely**
  - Then every **rearrangement** of Σa_n **converges to the same sum**

- Proof
  - Let $\Sigma a_n'$ be a rearrangement of Σa_n with partial sum s_n'
  - By the Cauchy Criterion, given $\varepsilon > 0, \exists N \in \mathbb{N}$ s.t.
    - $\sum_{i=n}^{m} |a_i| < \varepsilon, \forall m, n \geq N$
  - Choose p s.t. $1,2, \dots, N$ are all contained in the set $\{k_1, k_2, \dots, k_p\}$
  - Where $k_1, \dots, k_p$ are the indices of the rearranged series
  - Then if $n > p, a_1, \dots, a_N$ will be cancelled in the difference $s_n - s_n'$
  - So, $|s_n - s_n'| \leq \varepsilon \Rightarrow \{s_n'\}$ converges to the same value as $\{s_n\}$

Limit of Functions

Wednesday, April 11, 2018 12:15 PM

Definition 4.1: Limit of Functions

- Definition
  - Let X, Y be metric spaces, and $E \subset X$
  - Suppose $f: E \to Y$ and $\boldsymbol{p}$ **is a limit point of** $\boldsymbol{E}$
  - If $\exists q \in Y$ s.t.
    - $\boldsymbol{\forall \varepsilon > 0, \exists \delta > 0}$ **s.t.**
    - $\boldsymbol{0 < d_X(x, p) < \delta \Rightarrow d_Y(f(x), q) < \varepsilon}$
  - Then, we write $\boldsymbol{f(x) \to q}$ **as** $\boldsymbol{x \to p}$, or $\boldsymbol{\lim_{x \to p} f(x) = q}$
- Note
  - $0 < d_X(x, p) < \delta$ is the deleted neighborhood about p of radius δ
  - d_X and d_Y refer to the distances in X and Y, respectively
- Relationship with sequence
  - Theorem 4.2 relates this type of limit to the limit of a sequence
  - Consequently, if f has a limit at p, then its limit is unique

Definition 4.3: Algebra of Functions

- If $f: E \to \mathbb{R}^k, g: E \to \mathbb{R}^k$, then we define
- $(f + g)(x) = f(x) + g(x)$
- $(f - g)(x) = f(x) - g(x)$
- $(fg)(x) = f(x)g(x)$
- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ where $g(x) \neq 0$ on E

Theorem 4.4: Algebraic Limit Theorem of Functions

- Let X be a metric space, and $E \subset X$
- Suppose p be a limit point of E
- Let f, g be **complex functions** on E where
  - $\boldsymbol{\lim_{x \to p} f(x) = A}$ and $\boldsymbol{\lim_{x \to p} g(x) = B}$
- Then
  - $\boldsymbol{\lim_{x \to p} (f + g)(x) = A + B}$
  - $\boldsymbol{\lim_{x \to p} (f - g)(x) = A - B}$
  - $\boldsymbol{\lim_{x \to p} (fg)(x) = AB}$
  - $\boldsymbol{\lim_{x \to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}}$ where $\boldsymbol{B \neq 0}$

Continuous Function and Open Set

Monday, April 16, 2018 12:09 PM

Definition 4.5: Continuous Function

- Definition
  - Suppose X, Y are metric spaces, $E \subset X, p \in E$, and $f: E \to Y$
  - Then f is **continuous** at p if
    - For every $\varepsilon > 0$, there exists $\delta > 0$ s.t.
    - $x \in E, d_X(x,p) < \delta \Rightarrow d_Y\left(f(x), f(p)\right) < \varepsilon$
  - If f is continuous at every point $p \in E$, then f is **continuous on E**
- Note
  - **f must be defined at p** to be continous at p (as opposed to limit)
  - Every function is continuous at isolated point

Theorem 4.6: Continuity and Limits

- In the context of Definition 4.5, if **p is also a limit point of E**, then
- f is **continious** at p if and only if $\lim_{x \to p} f(x) = f(p)$

Theorem 4.7: Composition of Continuous Function

- Statement
  - Suppose X, Y, Z are metric spaces, $E \subset X, f: E \to Y, g: f(E) \to Z$, and
  - $h: E \to Z$ defined by $h(x) = g(f(x)), \forall x \in E$
  - If **f is continuous** at $p \in E$, and **g is continuous** at $f(p)$
  - Then **h is continuous** at p
- Note
  - h is called the composition of f and g and is written as $g \circ f$
- Proof
  - Let $\varepsilon > 0$ be given
  - Since $g: f(E) \to Z$ is continuous at $f(p), \exists \eta > 0$ s.t.
    - If $y \in f(E)$ and $d_Y\left(y, f(p)\right) < \eta$, then $d_Z\left(g(y), g\left(f(p)\right)\right) < \varepsilon$
  - Since $f: E \to Y$ is continuous at $p, \exists \delta > 0$ s.t.
    - If $x \in E$ and $d_X(x,p) < \delta$, then $d_Y\left(f(x), f(p)\right) < \eta$
  - Consequently, if $d_X(x,p) < \delta$, and $x \in E$, then
    - $d_Z\left(g(f(x)), g\left(f(p)\right)\right) = d_Z\left(h(x), h(p)\right) < \varepsilon$
  - So, h is continuous at p by definition

Theorem 4.8: Characterization of Continuity

- Statement
  - Given metric spaces X, Y
  - $f: X \to Y$ is **continuous** if and only if
  - $\boldsymbol{f^{-1}(V)}$ **is open** in X for **every open set** $\boldsymbol{V \subset Y}$
- Proof ($\Longrightarrow$)
  - Suppose f is continuous on X, and $V \subset Y$ is open
  - We want to show that all points of $f^{-1}(V)$ are interior points
  - Suppose $p \in X$, and $f(p) \in V$, then $p \in f^{-1}(V) \subset X$
  - Since V is open
    - There exists a neighborhood of $f(p)$ that is a subset of V
    - In other word, $\exists \varepsilon > 0$ s.t. $d_Y\left(y, f(p)\right) < \varepsilon \Rightarrow y \in V$
  - Since f is continuous at p
    - $\exists \delta > 0$ s.t. $d_X(x, p) < \delta \Rightarrow d_Y\left(f(x), f(p)\right) < \varepsilon$
  - Suppose $d_X(x, p) < \delta$
    - By the continuity of f, $d_Y\left(f(x), f(p)\right) < \varepsilon$
    - Then $f(x) \in V$, since V is open
    - Thus, $x \in f^{-1}(V)$
  - This shows that p is an interior point of $f^{-1}(V)$
  - Therefore $f^{-1}(V)$ is open in X
- Proof ($\Longleftarrow$)
  - Suppose $f^{-1}(V)$ is open in X for every open set $V \subset Y$
  - Let $p \in X$ and fix $\varepsilon > 0$
  - Let $V \coloneqq \left\{y \in Y \middle| d_Y\left(y, f(p)\right) < \varepsilon\right\}$ be the ε neighborhood of $f(p)$
  - Since V is open, $f^{-1}(V)$ is also open by assumption
  - Thus, $\exists \delta > 0$ s.t. $d_X(p, x) < \delta \Rightarrow x \in f^{-1}(V)$
  - But if $x \in f^{-1}(V)$, then $f(x) \in V$, and so $d_Y\left(f(x), f(p)\right) < \varepsilon$
  - So, $f: X \to Y$ is continuous at p
  - Since $p \in X$ was arbitrary, f is continuous on X
- Corollary
  - Given metric spaces X, Y
  - $f: X \to Y$ is **continuous** on X if and only if
  - $\boldsymbol{f^{-1}(V)}$ **is closed** in X for **every closed set** $\boldsymbol{V}$ **in** $\boldsymbol{Y}$
- Proof

- A set is closed if and only if its complement is open
- Also, $f^{-1}(E^c) = \left[f^{-1}(E)\right]^c$, for every $E \subset Y$

Continuity and Compactness, Extreme Value Theorem

Wednesday, April 18, 2018 12:06 PM

Definition 4.13: Boundedness

- A mapping $f: E \to \mathbb{R}^k$ is **bounded** if
- There is a real number M s.t. $\left|\boldsymbol{f(x)}\right| \leq \boldsymbol{M}, \forall \boldsymbol{x} \in \boldsymbol{E}$

Theorem 4.14: Continuous Functions Preserve Compactness

- Statement
  - Let X, Y be metric spaces, $\boldsymbol{X}$ **compact**
  - If $f: X \to Y$ is **continuous**, then $\boldsymbol{f(X)}$ **is also compact**
- Proof
  - Let $\{V_\alpha\}$ be an open cover of $f(X)$
  - f is continuous, so each of the sets $f^{-1}(V_\alpha)$ is open by Theorem 4.8
  - $\{f^{-1}(V_\alpha)\}$ is an open cover of X, and X is compact
  - So there is a finite set of indices $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ s.t.
    - $X \subset f^{-1}\left(V_{\alpha_1}\right) \cup f^{-1}\left(V_{\alpha_2}\right) \cup \cdots \cup f^{-1}\left(V_{\alpha_n}\right)$
  - Since $f\left(f^{-1}(E)\right) \subset E, \forall E \subset Y$
    - $f(X) \subset V_{\alpha_1} \cup V_{\alpha_2} \cup \cdots \cup V_{\alpha_n}$
  - This is a finite subcover of $f^{-1}(X)$

Theorem 4.15: Applying Theorem 4.14 to $\mathbb{R}^k$

- Statement
  - Let X be a **compact** metric space
  - If $f: X \to \mathbb{R}^k$ is **continuous**, then $f(X)$ is **closed** and **bounded**
  - Thus, f is **bounded**
- Proof
  - See Theorem 4.14 and Theorem 2.41

Theorem 4.16: Extreme Value Theorem

- Statement
  - Let f be a **continuous real function** on a **compact metric space** X
  - Let $M \coloneqq \sup_{p \in X} f(p)$, and $m \coloneqq \inf_{p \in X} f(p)$
  - Then $\exists p, q \in X$ s.t. $f(p) = M$ and $f(q) = m$
  - Equivalently, $\boldsymbol{\exists p, q \in X}$ **s.t.** $\boldsymbol{f(q) \leq f(x) \leq f(p), \forall x \in X}$
- Proof

- By Theorem 4.15, $f(X)$ is closed and bounded
- So $f(x)$ contains M and m by Theorem 2.28

Theorem 4.17: Inverse of Continuous Bijection is Continuous

- Statement
  - Let X, Y be metric spaces, $\boldsymbol{X}$ **compact**
  - Suppose $f: X \to Y$ is **continuous** and **bijective**
  - Define $f^{-1}: Y \to X$ by $f^{-1}\big(f(x)\big) = x, \forall x \in X$
  - Then $\boldsymbol{f^{-1}}$ is also **continuous** and **bijective**
- Proof
  - By Theorem 4.8, it suffices to show $f(V)$ is open in Y for all open sets $V \subset X$
  - Fix an open set V in X
  - V is open in compact metric space X
  - So V^c is closed and compact by Theorem 2.35
  - Therefore, $f(V^c)$ is a compact subset of Y by Theorem 4.14
  - So $f(V^c)$ is closed in Y by Theorem 2.34
  - f is 1-1 and onto, so $f(V) = \big(f(V^c)\big)^c$
  - Therefore $f(V)$ is open

Uniform Continuity and Compactness

Friday, April 20, 2018 12:10 PM

Definition 4.18: Uniform Continuity

- Let X, Y be metric spaces, $f: X \to Y$
- f is **uniformly continuous** on X if $\forall \varepsilon > 0, \exists \delta > 0$ s.t.
- If $\boldsymbol{p, q \in X}$ and $\boldsymbol{d_X(p,q) < \delta}$, then $\boldsymbol{d_Y\left(f(p), f(q)\right) < \varepsilon}$

Theorem 4.19: Uniform Continuity and Compactness

- Statement
  - Let X, Y be metric spaces, $\boldsymbol{X}$ **compact**
  - If $f: X \to Y$ is **continuous**, then f is also **uniformly continuous**
- Proof
  - Let $\varepsilon > 0$ be given
  - Since f is continuous, $\forall p \in X, \exists \phi(p)$ s.t.
    - If $q \in X$, and $d_X(p,q) < \phi(p)$, then $d_Y\left(f(p), f(q)\right) < \frac{\varepsilon}{2}$
  - Let $J(p) \coloneqq \left\{q \in X \middle| d_X(p,q) < \frac{1}{2}\phi(p)\right\}$
    - $p \in J(p), \forall p \in X$, so $\{J(p)\}$ is an open cover of X
    - Since X is compact, $\{J(p)\}$ has a finite subcover
    - So there exists finite set of points $p_1, \dots, p_n \in X$ s.t.
    - $X \subset J(p_1) \cup \dots \cup J(p_n)$
  - Let $\delta = \frac{1}{2}\min\{\phi(p_1), \dots, \phi(p_n)\} > 0$
  - Let $p, q \in X$ s.t. $d_X(p,q) < \delta$
    - Since $X \subset J(p_1) \cup \dots \cup J(p_n)$,
    - $\exists m \in \{1,2,\dots,n\}$ s.t. $p \in J(p_m)$
  - Hence,
    - $d_X(p, p_m) < \frac{1}{2}\phi(p_m) < \phi(p_m)$
    - $d_X(q, p_m) \le d_X(p,q) + d_X(p, p_m) < \delta + \frac{1}{2}\phi(p_m) \le \phi(p_m)$
  - By the triangle inequality and definition of $\phi(p)$,
    - $d_Y\left(f(p), f(q)\right) \le d_Y\left(f(p), f(p_m)\right) + d_Y\left(f(p_m), f(q)\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$
  - Therefore f is uniformly continuous

Theorem 4.20: Continuous Mapping from Noncompact Set

- Definition
  - Let E be **noncompact** set in $\mathbb{R}$
  - Then there exists a continuous function f on E s.t.
    - (a) f is **not bounded**
    - (b) f is bounded but **has no maximum**
    - (c) ***E* is bounded**, but f is **not uniformly continuous**
- Proof : If E is bounded
  - Since E is noncompact, E must be not closed
  - So there exists a limit point $x_0 \in E$ s.t. $x_0 \notin E$
  - $f(x) \coloneqq \dfrac{1}{x - x_0}$ establishes (c)
    - f is continuous by Theorem 4.9
    - f is clearly unbounded
    - f is not uniformly continuous
      - Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary
      - Choose $x \in E$ s.t. $|x - x_0| < \delta$
      - Taking t close to x_0
      - We can make $|f(t) - f(x)| > \varepsilon$, but $|t - x| < \delta$
      - Since $\delta > 0$ is arbitrary
  - $g(x) \coloneqq \dfrac{1}{1 + (x - x_0)^2}$ establishes (b)
    - g is continuous by Theorem 4.9
    - g is bounded, since $0 < g(x) < 1$
    - g has no maximum, since $\sup_{x \in E} g(x) = 1$, but $g(x) < 1$
- Proof: If E is not bounded
  - $f(x) \coloneqq x$ establishes (a)
  - $h(x) \coloneqq \dfrac{x^2}{1 + x^2}$ establishes (b)

Example 4.21: Inverse Mapping and Noncompact Set

- Let $X = [0,2\pi)$
- Let $f: X \to Y$ given by $f(t) = (\cos t, \sin t)$
- Then f is continuous, and bijective
- But f^{-1} is not continuous at $f(0) = (1,0)$

Connected Set, Intermediate Value Theorem

Monday, April 23, 2018 12:10 PM

Definition 2.45: Connected Set

- Let X be a metric space, and $A, B \subset X$
- A and B are **separated** if
  - $A \cup \bar{B} = \emptyset$ and $\bar{A} \cup B = \emptyset$
  - i.e. No point of A lies in the closure of B and vice versa
- $E \subset X$ is **connected** if
  - E is **not** a union of **two nonempty separated sets**

Theorem 2.47: Connected Subset of $\mathbb{R}$

- Statement
  - $\boldsymbol{E} \subset \mathbb{R}$ is **connected** if and only if E has the following property
  - If $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{E}$ and $\boldsymbol{x} < \boldsymbol{z} < \boldsymbol{y}$, then $\boldsymbol{z} \in \boldsymbol{E}$
- Proof ($\Longrightarrow$)
  - By way of contrapositive, suppose $\exists x, y \in E$, and $z \in (x, y)$ s.t. $z \notin E$
  - Let $A_z = E \cap (-\infty, z)$ and $B_z = E \cap (z, +\infty)$
  - Then A_z and B_z are separated and $E = A_z \cup B_z$
  - Therefore E is not connected
- Proof ($\Longleftarrow$)
  - By way of contrapositive, suppose E is not connected
  - Then there are nonempty separated sets A and B s.t. $E = A \cup B$
  - Let $x \in A, y \in B$. Without loss of generality, assume $x < y$
  - Let $z \coloneqq \sup(A \cap [x, y])$. Then by Theorem 2.28, $z \in \bar{A}$
  - By definition of E, $z \notin B$. So, $x \leq z < y$
  - If $z \notin A$
    - $x \in A$ and $z \notin A$
    - $\Rightarrow x < z < y$
    - $\Rightarrow z \notin E$
  - If $z \in A$
    - Since A and B are separated, $z \notin \bar{B}$
    - So $\exists z_1$ s.t. $z < z_1 < y$ and $z_1 \notin B$
    - Then $x < z_1 < y$, so $z_1 \notin E$

Theorem 4.22: Continuous Mapping of Connected Set

- Statement

- Let X, Y be metric spaces
- Let $f: X \to Y$ be a **continuous mapping**
- If $\boldsymbol{E \subset X}$ **is connected** then $\boldsymbol{f(E) \subset Y}$ **is also connected**

- Proof
  - Suppose, by way of contradiction, that $f(E)$ is not connected
  - i.e. $f(E) = A \cup B$, where $A, B \subset Y$ are nonempty and separated
  - Let $G \coloneqq E \cap f^{-1}(A)$ and $H \coloneqq E \cap f^{-1}(B)$
  - Then $E = G \cup H$, where $G, H \neq \emptyset$
  - Since $A \subset \bar{A}$, we have $G \subset f^{-1}(\bar{A})$
  - Since f is continuous and $\bar{A}$ is closed, $f^{-1}(\bar{A})$ is also closed
  - Therefore $\bar{G} \subset f^{-1}(\bar{A})$, and hence $f(\bar{G}) \subset \bar{A}$
  - Since $f(H) = B$ and $\bar{A} \cap B = \emptyset$, we have $\bar{G} \cap H = \emptyset$
  - Similarly, $G \cap \bar{H} = \emptyset$
  - So, G and H are separated
  - This is a contradiction, therefore $f(E)$ is connected

Theorem 4.23: Intermediate Value Theorem

- Statement
  - Let $f: \mathbb{R} \to \mathbb{R}$ be **continuous** on $[a, b]$
  - If $\boldsymbol{f(a) < f(b)}$ and if c statifies $\boldsymbol{f(a) < c < f(b)}$
  - Then $\exists \boldsymbol{x} \in \boldsymbol{(a, b)}$ **s.t.** $\boldsymbol{f(x) = c}$
- Proof
  - By Theorem 2.47, $[a, b]$ is connected
  - By Theorem 4.22, $f([a, b])$ is a connected subset of $\mathbb{R}$
  - By Theorem 2.47, the result follows

Derivative, Chain Rule, Local Extrema

Wednesday, April 25, 2018 12:19 PM

Definition 5.1: Derivative

- Let f be defined (and real-valued) on $[a, b]$
- $\forall x \in [a, b]$, let $\phi(t) = \dfrac{f(t) - f(x)}{t - x}$ $(a < t < b, t \neq x)$
- Define $\boldsymbol{f'(x) = \lim_{t \to x} \phi(t)}$, provided that this limit exists
- f' is called the **derivative** of f
- If f' is defined at point x, f is **differentiable** at x
- If f' is defined $\forall x \in E \subset [a, b]$, then f is differentiable on E

Theorem 5.2: Differentiability Implies Continuity

- Statement
  - Let f be defined on $[a, b]$
  - If f is **differentiable** at $x \in [a, b]$ then f is **continuous** at x
- Proof
  - $\lim_{t \to x}\big(f(t) - f(x)\big) = \lim_{t \to x}\left(\dfrac{f(t) - f(x)}{t - x}(t - x)\right) = \lim_{t \to x}\big(f'(x)(t - x)\big) = 0$
  - So $\lim_{t \to x} f(t) = f(x)$

Theorem 5.5: Chain Rule

- Statement
  - Given
    - f is **continuous** on $[a, b]$, and $\boldsymbol{f'(x)}$ **exists** at $x \in [a, b]$
    - g is defined on $I \supset \text{im}(f)$, and $\boldsymbol{g}$ **is differentiable at** $\boldsymbol{f(x)}$
  - If $\boldsymbol{h(t) = g(f(t))}$ $(a \leq t \leq b)$, then
    - $\boldsymbol{h}$ **is differentiable at** $\boldsymbol{x}$, and $\boldsymbol{h'(x) = g'(f(x)) \cdot f'(x)}$
- Proof
  - Let $y = f(x)$
  - By the definition of derivative
    - $f(t) - f(x) = (t - x)\big(f'(x) + u(t)\big)$, where $t \in [a, b]$, $\lim_{t \to x} u(t) = 0$
    - $g(s) - g(y) = (s - y)\big(g'(y) + v(s)\big)$, where $s \in I$, $\lim_{s \to y} v(s) = 0$
  - Let $s = f(t)$, then
    - $h(t) - h(x)$
    - $= g\big(f(t)\big) - g\big(f(x)\big)$

        - $= (f(t) - f(x))\left(g'(y) + v(s)\right)$
        - $= (t - x)(f'(x) + u(t))\left(g'(y) + v(s)\right)$
    - If $t \neq x$, then
        - $\dfrac{h(t) - h(x)}{t - x} = (f'(x) + u(t))\left(g'(y) + v(s)\right)$
    - As $t \to x$
        - $u(t) \to 0$, and $v(s) \to 0$
        - So $s = f(t) \to f(x) = y$ by continuity
    - Therefore $h'(x) = \lim_{t \to x} \dfrac{h(t) - h(x)}{t - x} = f'(x)g'(y) = g'(f(x))f'(x)$

Definition 5.7: Local Maximum and Local Minimum

- Let X be a metric space, $f: X \to \mathbb{R}$
- f has a **local maximum** at $p \in X$ if $\exists \delta > 0$ s.t.
    - $\boldsymbol{f(q) \leq f(p)}, \forall q \in X$ s.t. $\boldsymbol{d(p, q) < \delta}$
- f has a **local minimum** at $p \in X$ if $\exists \delta > 0$ s.t.
    - $\boldsymbol{f(q) \geq f(p)}, \forall q \in X$ s.t. $\boldsymbol{d(p, q) < \delta}$

Theorem 5.8: Local Extrema and Derivative

- Statement
    - Let f be defined on $[a, b]$
    - If f has a **local maximum (or minimum)** at $x \in (a, b)$
    - Then $\boldsymbol{f'(x) = 0}$ if it exists
- Proof
    - By Definition 5.7, choose δ, then
        - $a < x - \delta < x < x + \delta < b$
    - Suppose $x - \delta < t < x$
        - $\dfrac{f(t) - f(x)}{t - x} \geq 0$
        - Let $t \to x$ (with $t < x$), then $f'(x) \geq 0$
    - Suppose $x < t < x + \delta$
        - $\dfrac{f(t) - f(x)}{t - x} \leq 0$
        - Let $t \to x$ (with $t > x$), then $f'(x) \leq 0$
    - Therefore $f'(x) = 0$

Mean Value Theorem, Monotonicity, Taylor's Theorem

Friday, April 27, 2018 12:07 PM

Theorem 5.9: Extended Mean Value Theorem

- Statement
  - Given
    - f and g are **continuous** real-valued functions on $[a, b]$
    - f, g are **differentiable** on (a, b)
  - Then there is a point $x \in (a, b)$ at which
    - $\boldsymbol{[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)}$
- Proof
  - Let $h(t) \coloneqq [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t),\ (a \leq t \leq b)$
  - Then h is continuous on $[a, b]$ and differentiable on (a, b)
  - We want to show that $h'(x) = 0$ for some $x \in (a, b)$
  - By definition of h, we have $h(a) = f(b)g(a) - f(a)g(b) = h(b)$
  - If h is constant
    - $h'(x) = 0$ on all of (a, b), and we are done
  - If h is not constant
    - $\exists t \in (a, b)$ s.t. $h(t) > h(a) = h(b)$ or $h(t) < h(a) = h(b)$
    - By Theorem 4.16, $\exists x \in (a, b)$ s.t.
    - $h(x)$ is either a global maximum or a global minimum
    - By Theorem 5.8, $h'(x) = 0$

Theorem 5.10: Mean Value Theorem

- Statement
  - Let $f: [a, b] \to \mathbb{R}$
  - If f is **continuous on** $[\boldsymbol{a}, \boldsymbol{b}]$ and **differentiable on** $(\boldsymbol{a}, \boldsymbol{b})$
  - Then $\exists x \in (a, b)$ s.t. $f(b) - f(a) = (b - a)f'(x)$
- Proof
  - Let $g(x) = x$ in Theorem 5.9

Theorem 5.11: Derivative and Monotonicity

- Suppose f is differentiable on (a, b)
- If $\boldsymbol{f'(x) \geq 0}, \forall x \in (a, b)$, then f is **monotonically increasing**
- If $\boldsymbol{f'(x) = 0}, \forall x \in (a, b)$, then f is **constant**
- If $\boldsymbol{f'(x) \leq 0}, \forall x \in (a, b)$, then f is **monotonically decreasing**

Theorem 5.15: Taylor's Theorem

- Statement
  - Suppose
    - f is a real-valued function on $[a, b]$
    - Fix a positive integer n
    - $\boldsymbol{f^{(n-1)}}$ **is continuous on** $(\boldsymbol{a}, \boldsymbol{b})$
    - $\boldsymbol{f^{(n)}(t)}$ **exists** $\forall \boldsymbol{t} \in (\boldsymbol{a}, \boldsymbol{b})$
  - Let $\alpha, \beta \in [a, b]$, where $a \neq \beta$
  - Define $\boldsymbol{P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k}$
  - Then $\exists \boldsymbol{x}$ **between** $\boldsymbol{\alpha}$ **and** $\boldsymbol{\beta}$ s.t.
  - $\boldsymbol{f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n}$
- Note
  - When $n = 1$, this is the Meal Value Theorem
- Proof
  - Without loss of generality, suppose $\alpha < \beta$
  - Define $M \in \mathbb{R}$ by
    - $f(\beta) = P(\beta) + M(\beta - \alpha)^n$
  - Then we want to show that
    - $n!\,M = f^{(n)}(x)$ for some $x \in [\alpha, \beta]$
  - Define difference function g by
    - $g(t) = f(t) - P(t) - M(t-\alpha)^n$, where $a \leq t \leq b$
    - Then $g(\beta) = 0$ by our choice of M
    - Taking derivative n times on both side, we get
    - $g^{(n)}(t) = f^{(n)}(t) - n!\,M$, where $a \leq t \leq b$
    - Note that $P(t)$ disappears, since its degree is $n - 1$
  - Now we only need to show $g^{(n)}(x) = 0$ for some $x \in [\alpha, \beta]$
    - $P^{(k)}(\alpha) = f^{(k)}(\alpha)$, for $0 \leq k \leq n-1$, by definition of P
    - Therefore, $g(\alpha) = g'(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0$
    - Also, $g(\beta) = 0$, by definition of M
    - By the Mean Value Theorem, $g'(x_1) = 0$ for some $x_1 \in [\alpha, \beta]$
    - $g'(\alpha) = 0$, so $g''(x_2) = 0$ for some $x_2 \in [\alpha, x_1]$
    - After n steps, $g^{(n)}(x_n) = 0$ for some $x_n \in [\alpha, x_{n-1}]$
    - So, $x_n \in [\alpha, \beta]$

Riemann-Stieltjes Integral, Refinement

Monday, April 30, 2018 12:12 PM

Definition 6.1: Riemann Integral

- Partition
  - A **partition** P of a closed interval $[a, b]$ is a **finite** set of points
  - $\{x_0, x_1, \dots, x_n\}$ where $\boldsymbol{a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b}$
- Let f be a bounded real function on $[a, b]$, for each partition P of $[a, b]$
  - Define M_i and m_i to be
    - $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$
    - $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$
  - Define the **upper sum** and **lower sum** to be
    - $U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$
    - $L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$
    - where $\Delta x_i = x_i - x_{i-1}$
  - Define the **upper and lower Reimann integral** to be
    - $\overline{\int_a^b} f dx = \inf_{\text{All } P} U(P, f)$
    - $\underline{\int_a^b} f dx = \sup_{\text{All } P} L(P, f)$
- If $\overline{\int_a^b} f dx = \underline{\int_a^b} f dx$, then
  - We say that f is Riemann-integrable on $[a, b]$, and write $f \in \mathcal{R}$
  - Their common value is denoted by $\int_a^b f dx$ or $\int_a^b f(x) dx$
- Well-definedness of upper and lower Riemann integral
  - Since f is bounded, $\exists m, M \in \mathbb{R}$ s.t.
    - $m \leq f(x) \leq M \ (a \leq x \leq b)$
  - Therefore for every partition P of $[a, b]$
    - $m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a)$
  - So $\overline{\int_a^b} f dx$ and $\underline{\int_a^b} f dx$ are always defined

Definition 6.2: Riemann-Stieltjes Integral

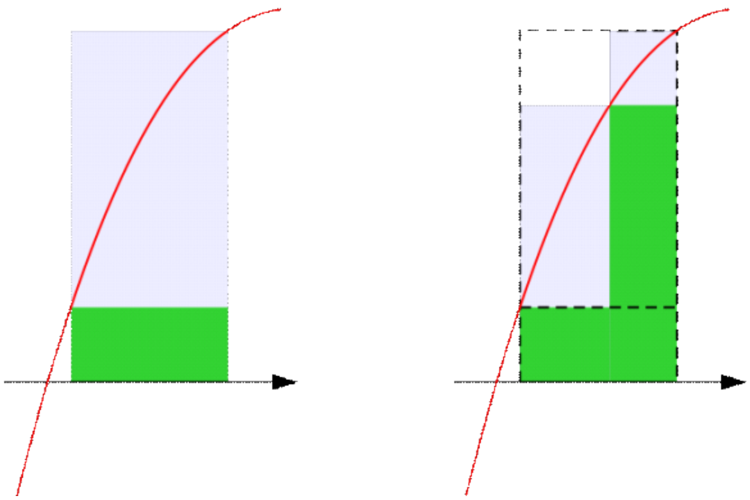
- Let α be a monotonically increasing function on $[a, b]$
- Let f be a real-valued function bouned on $[a, b]$
- For each partition P of $[a, b]$, define
  - $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$
  - $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$
  - $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$
  - $U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$
  - $L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$
  - $\overline{\int_a^b} f dx = \inf_{\text{All } P} U(P, f, \alpha)$
  - $\underline{\int_a^b} f dx = \sup_{\text{All } P} L(P, f, \alpha)$
- If $\overline{\int_a^b} f dx = \underline{\int_a^b} f dx$
  - We denote the common value by $\int_a^b f d\alpha$ or $\int_a^b f(x) d\alpha(x)$
  - This is the **Riemann-Stieltjes integral** of f with respect to α over $[a, b]$
  - We say f is integrable with respect to α with on $[a, b]$, and write $f \in \mathcal{R}(\alpha)$
- Note
  - When $\alpha(x) = x$, this is just Riemann integral

Definition 6.3: Refinement and Common Refinement

- We say that the partition P^* is a **refinement** of P if $\boldsymbol{P^* \supset P}$
- Given two partitions P_1 and P_2, their **common refinement** is $P_1 \cup P_2$

Theorem 6.4: Properties of Refinement

- If P^* is a refinement of P, then
- $\boldsymbol{L(P, f, \alpha) \leq L(P^*, f, \alpha)}$
- $\boldsymbol{U(P^*, f, \alpha) \leq U(P, f, \alpha)}$



Theorem 6.5: Properties of Common Refinement

- Statement
  - $\overline{\int_a^b} \boldsymbol{f\,dx} \leq \underline{\int_a^b} \boldsymbol{f\,dx}$
- Proof Outline
  - Given 2 partitions P_1 and P_2
  - Let P^* be the common refinement
  - Then $L(P_1, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P_2, f, \alpha)$

Theorem 6.6

- Statement
  - $\boldsymbol{f} \in \boldsymbol{\mathcal{R}(\alpha)}$ **on** $[\boldsymbol{a}, \boldsymbol{b}]$ if and only if
  - $\forall \varepsilon > 0$, there exists a partition P s.t. $\boldsymbol{U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon}$
- Proof Outline
  - $\forall P, L(P, f, \alpha) \leq \underline{\int_a^b} f\,dx \leq \overline{\int_a^b} f\,dx \leq U(P, f, \alpha)$
  - $(\Leftarrow)$ If $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$
    - Then $0 \leq \overline{\int_a^b} f\,dx - \underline{\int_a^b} f\,dx < \varepsilon$
  - $(\Rightarrow)$ If $f \in \mathcal{R}(\alpha)$
    - Then $\exists P_1, P_2$ s.t.
      - $U(P_1, f, \alpha) - \int_a^b f\,d\alpha < \frac{\varepsilon}{2}$
      - $\int_a^b f\,d\alpha - L(P_1, f, \alpha) < \frac{\varepsilon}{2}$
    - Consider their common refinement P

- By Theorem 6.4, $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

Theorem 6.8

- If f is **continuous** on $[a, b]$, then $\boldsymbol{f \in \mathcal{R}(\alpha)}$ on $[a, b]$

Theorem 6.9

- If f is **monotonic** on $[a, b]$, and α is **continuous** on $[a, b]$
- Then $\boldsymbol{f \in \mathcal{R}(\alpha)}$ on $[a, b]$

Theorem 6.10

- If f is **bounded** on $[a, b]$ with **finitely many points of discontiunity**
- And $\boldsymbol{\alpha}$ **is continuous** on these points, then $\boldsymbol{f \in \mathcal{R}(\alpha)}$

Fundamental Theorem of Calculus

May 2, 2018 12:11 PM

Final Exam

- Thursday, May 10, 7:45 - 9:45 AM, @ Social Science 6102
- 5 or 6 questions
  - ~1 question from Exam 1 / Quiz
  - ~1 question from Exam 2
  - ~1 question on absolute convergence and/or power series
  - ~1 question on continuity
  - ~1 question on derivatives and/or integrals
  - Nothing from Chapter 7

Theorem 6.20: Fundamental Theorem of Calculus (Part I)

- Statement
  - Let $f \in \mathcal{R}$ on $[a, b]$
  - Define $F(x) = \int_a^x f(t)dt$ for $x \in [a, b]$, then
    - $\boldsymbol{F}$ **is continuous** on $[a, b]$
  - Furthermore, if $\boldsymbol{f}$ **is continuous at** $\boldsymbol{x_0} \in [a, b]$, then
    - $\boldsymbol{F}$ **is differentiable at** $\boldsymbol{x_0}$, and
    - $\boldsymbol{F'(x_0) = f(x_0)}$
- Proof: F is continuous on $[a, b]$
  - Since $f \in \mathcal{R}$, f is bounded, so $\exists M \in \mathbb{R}$ s.t.
    - $|f(t)| \leq M, \forall a \leq t \leq b$
  - If $a \leq x < y \leq b$, then
    - $|F(y) - F(x)| = \left|\int_y^x f(t)dt\right| \leq M(x - y)$
  - Given $\varepsilon > 0$
    - $|F(y) - F(x)| < \varepsilon$ provided $|y - x| < \frac{\varepsilon}{M}$
  - So this shows **uniform continuity** of F
- Proof: $F'(x_0) = f(x_0)$
  - Suppose f is continuous at x_0
  - Given $\varepsilon > 0, \exists \delta > 0$ s.t.
    - $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$ for $a \leq x \leq b$
  - If $x_0 - \delta < s \leq x_0 \leq t < x_0 + \delta$ where $a \leq s < t \leq b$, then

- $\left|\frac{F(t)-F(s)}{t-s}-f(x_0)\right|$
- $=\left|\left(\frac{1}{t-s}\int_s^t f(x)dx\right)-f(x_0)\right|$
- $=\left|\left(\frac{1}{t-s}\int_s^t f(x)dx\right)-\left(\frac{1}{t-s}\int_s^t f(x_0)dx\right)\right|$
- $=\left|\frac{1}{t-s}\int_s^t \left(f(x)-f(x_0)\right)dx\right|$
- $<\left|\frac{1}{t-s}(t-s)\varepsilon\right|=\varepsilon$
- Consequently, $F'(x_0)=f(x_0)$

Theorem 6.21: Fundamental Theorem of Calculus (Part II)

- Statement
  - Let $f\in\mathcal{R}$ on $[a,b]$
  - If there exists a **differentiable function** $\boldsymbol{F}$ on $[a,b]$ s.t. $\boldsymbol{F}'=\boldsymbol{f}$
  - Then $\int_{\boldsymbol{a}}^{\boldsymbol{b}}\boldsymbol{f(x)dx}=\boldsymbol{F(b)}-\boldsymbol{F(a)}$
- Proof
  - Let $\varepsilon>0$ be given
  - Choose a partition $P=\{x_0,x_1,\dots,x_n\}$ of $[a,b]$ s.t.
    - $U(P,f)-L(P,f)<\varepsilon$
  - Apply the Meal Value Theorem, $\exists t_i\in[x_{i-1},x_i]$ s.t.
    - $F(x_i)-F(x_{i-1})=f(t_i)\Delta x_i$ where $1\le i\le n$
  - Thus, $\sum_{i=1}^{n}f(t_i)\Delta x_i$ forms a telescoping series
    - $\sum_{i=1}^{n}f(t_i)\Delta x_i=F(x_n)-F(x_{n-1})+F(x_{n-1})+\cdots-F(x_0)$
    - $=F(b)+\left(F(x_{n-1})-F(x_{n-1})\right)+\cdots+\left(F(x_1)-F(x_1)\right)-F(a)$
    - $=F(b)-F(a)$
  - Combining the obvious inequalities below
    - $L(P,f)\le\sum_{i=1}^{n}f(t_i)\Delta x_i\le U(P,f)$
    - $L(P,f)\le\int_a^b f dx\le U(P,f)$
  - We get

- $\left|\sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f\,dx\right| < \varepsilon$
- $\Rightarrow \left|F(b) - F(a) - \int_a^b f\,dx\right| < \varepsilon$

- Therefore, $\int_a^b f(x)dx = F(b) - F(a)$

Sequence of Functions, Uniform Convergence

May 4, 2018 12:10 PM

Definition 7.1: Limit of Sequence of Functions

- Suppose $\{f_n\}$ is a **sequence of functions** defined on a set E
- Suppose the **sequence of numbers** $\{\boldsymbol{f_n(x)}\}$ **converges** $\forall x \in E$
- We can then defined f by $\boldsymbol{f(x) = \lim_{n\to\infty} f_n(x)}$, $\forall x \in E$

Example 7.2: Double Sequence

- Let $s_{m,n} = \dfrac{m}{m+n}$, $(m, n \in \mathbb{N})$
- Fix $n \in \mathbb{N}$
  - $\lim_{m\to\infty} s_{m,n} = 1$
  - $\lim_{n\to\infty} \lim_{m\to\infty} s_{m,n} = 1$
- Fix $m \in \mathbb{N}$
  - $\lim_{n\to\infty} s_{m,n} = 0$
  - $\lim_{m\to\infty} \lim_{n\to\infty} s_{m,n} = 0$

Example 7.3: Convergent Series of Continuous Functions

- Let $f_n(x) = \dfrac{x^2}{(1+x^2)^n}$, $(x \in \mathbb{R}, n \in \mathbb{Z}_{\geq 0})$
- Let $f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \dfrac{x^2}{(1+x^2)^n}$
- When $x = 0$
  - $f_n(0) = 0$, so $f(0) = 0$
- When $x \neq 0$
  - $f(x)$ is a convergent geometric series with sum
  - $f(x) = \sum_{n=0}^{\infty} \dfrac{x^2}{(1+x^2)^n} = \dfrac{x^2}{1 - \left(\dfrac{1}{1+x^2}\right)^n} = 1 + x^2$
- Therefore, $f(x) = \begin{cases} 0 & \text{for } x = 0 \\ 1 + x^2 & \text{for } x \neq 0 \end{cases}$
- So **convergent series of continuous functions may be discontinuous**

Example 7.5: Changing the Order of Limit and Derivative

- Let $f_n(x) = \dfrac{\sin(nx)}{\sqrt{n}}$, $(x \in \mathbb{R}, n \in \mathbb{N})$
- Let $f(x) = \lim_{n\to\infty} f_n(x) = 0$

- Then $f'(x) = 0$, but $f_n'(x) = \sqrt{n}\cos(nx) \to \infty \neq 0$

Example 7.6: Changing the Order of Limit and Integral

- Let $f_n(x) = nx\left(1 - x^2\right)^n$, $(x \in [0,1], n \in \mathbb{N})$, then
- $\lim_{n\to\infty}\left(\int_0^1 f_n(x)dx\right) = \lim_{n\to\infty}\left(\int_0^1 nx\left(1-x^2\right)^n dx\right) = \lim_{n\to\infty}\frac{n}{2n+2} = \frac{1}{2}$
- $\int_0^1 \left(\lim_{n\to\infty} f_n(x)\right)dx = \int_0^1 \left(\lim_{n\to\infty} nx\left(1-x^2\right)^n\right)dx = \int_0^1 0\,dx = 0$

Definition 7.7: Uniform Convergence

- A sequence of function $\{f_n\}_{n\in\mathbb{N}}$ **converges uniformly** on E to a function f if
- $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. if $n \geq N$, then $\boldsymbol{|f_n(x) - f(x)| < \varepsilon}, \forall x \in E$

Theorem 7.11: Interchange of Limits

- Suppose $f_n \to f$ on a set E **uniformly** on a metric space
- Let x be a limit point of E and suppose that $\lim_{t\to x} f_n(t) = A_n, (n \in \mathbb{N})$
- Then $\{A_n\}$ converges and $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$
- i.e. $\boldsymbol{\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)}$

Theorem 7.12: Uniform Convergence Implies Continuity

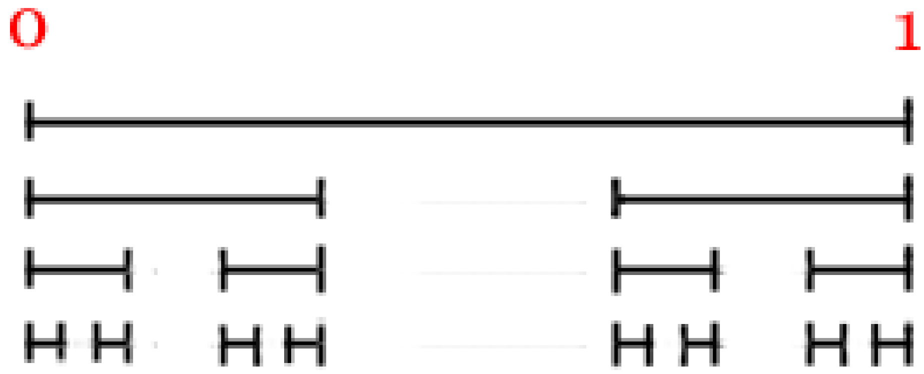
- If $\{f_n\}$ is a sequence of **continuous functions** on E, and $\boldsymbol{f_n \to f}$ **uniformly** on E
- Then f is **continuous** on E

Definition 7.14: Space of Bounded Continuous Functions

- Let X be a metric space
- Let $\mathcal{C}(X)$ be the set of **all continuous bounded functions** $f: X \to \mathbb{C}$
- If $f \in \mathcal{C}(X)$, define the s**upremum norm** $\boldsymbol{\|f\| \coloneqq \sup_{x\in X}|f(x)|}$
- $\|f - g\|$ is a distance function that makes $\boldsymbol{\mathcal{C}(X)}$ **a metric space**

Example 2.44: Cantor Set

- Define a sequence of compact sets E_n
  - $E_0 = [0,1]$
  - $E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$
  - $E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$
  - $\vdots$
- The set $P \coloneqq \bigcap_{n=1}^{\infty} E_n$ is called the **Cantor Set**



- P is **compact**, **nonempty**, **uncountable**, **perfect**, **measure zero**

Example 4.27: Discontinuous Function

- Let $f(x) \coloneqq \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$
- Then $f(x)$ is discontinuous at all $x \in \mathbb{R}$
- Let $g(x) \coloneqq \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$
- Then $g(x)$ is discontinuous everywhere except $x = 0$