Math 632 004 HW01

Shawn Zhong

TOTAL POINTS

80 / 80

QUESTION 1

1 10 pts

1.1 a **4 / 4**

✓ **- 0 pts** Correct

1.2 b **3 / 3**

✓ **- 0 pts** Correct

1.3 c **3 / 3**

✓ **- 0 pts** Correct

QUESTION 2

2 2 **10 / 10**

✓ **- 0 pts** Correct

QUESTION 3

3 10 pts

3.1 a **5 / 5**

✓ **- 0 pts** Correct

3.2 b **5 / 5**

✓ **- 0 pts** Correct

QUESTION 4

4 Textbook 1.1 **10 / 10**

✓ **- 0 pts** Correct

QUESTION 5

5 Textbook 1.2 **10 / 10**

✓ **- 0 pts** Correct

QUESTION 6

6 Textbook 1.5 **10 / 10**

✓ **- 0 pts** Correct

QUESTION 7

Textbook 1.6 10 pts

7.1 a **4 / 4**

✓ **- 0 pts** Correct

7.2 b **6 / 6**

✓ **- 0 pts** Correct

QUESTION 8

Textbook 1.7 10 pts

8.1 a **4 / 4**

✓ **- 0 pts** Correct

8.2 b **3 / 3**

✓ **- 0 pts** Correct

8.3 c **3 / 3**

✓ **- 0 pts** Correct

HW1 - Problem & Answer

Thursday, September 6, 2018 2:57 PM

Math 632 Lecture 4, Fall 2018, Homework 1

Due Tuesday, September 18, 9:30am

This homework set has eight problems: first three problems to review some basic probability, and then five problems from Durrett's book.

Please check the homework instructions on the course homepage. In particular:

- Credit comes from your reasoning, not your numerical answer.
- Observe rules of academic integrity. You are encouraged to discuss the problems with fellow students, but copied work is not acceptable and will result in zero credit.
- You may find solutions to some exercises on the web. However, the point of the homework is to give you the problem solving practice you need in the exams. Hence it is not smart to take shortcuts to secure the few points that come from homework.

1. Let A and B be two events on the same sample space.

   (a) Suppose $P(A) = p$, $P(B) = q$, and $P(A|B) = r$. Deduce $P(A \cap B)$ and $P(B|A)$.

   (b) Give an example where $P(A|B) > P(A)$ and neither A nor B is equal to the whole space or the empty set. (This means that you give a sample space Ω, a probability measure P on Ω, and define two events A and B on Ω that satisfy the question.)

   (c) Give an example where $P(A|B) < P(A)$ and neither A nor B is equal to the whole space or the empty set.

(a) $P(A \cap B) = P(A|B) \cdot P(B) = rq$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{rq}{p}$$

(b) $\Omega = \{(1,2), (1,3), (1,4), (2,3), (2,4)\}$

$A = \{(x,y) \mid x = 2\}$

1.1 a **4 / 4**

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   (a) Suppose $P(A) = p$, $P(B) = q$, and $P(A|B) = r$. Deduce $P(A \cap B)$ and $P(B|A)$.
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(a) $P(A \cap B) = P(A|B) \cdot P(B) = rq$

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(b) $\Omega = \{(1,2), (1,3), (1,4), (2,3), (2,4)\}$

$A = \{(x,y) \mid x = 2\}$

$B = \{ (x,y) \mid y \geq 3 \}$

Then $P(A) = 0.4$, $P(A|B) = 0.5$

(c) $\Omega = \{ (1,2), (1,3), (1,4), (2,3), (2,4) \}$

$A = \{ (x,y) \mid x = 1 \}$

$B = \{ (x,y) \mid y \geq 3 \}$

Then $P(A) = 0.6$, $P(A|B) = 0.5$

2. Let p, q, r be positive reals such that $p + q + r = 1$ and let n be a positive integer. Let (X, Y, Z) have multinomial distribution with parameters $(n, 3, p, q, r)$. This means that the joint probability mass function is

$$P(X = k, Y = \ell, Z = m) = \frac{n!}{k!\,\ell!\,m!} p^k q^\ell r^m$$

for integers $k, \ell, m \geq 0$ such that $k+\ell+m = n$. This distribution arises from the following experiment: perform n independent repetitions of a trial with three possible outcomes with probabilities p, q, r and let X, Y, Z count how many times these outcomes appear.

Calculate the probability mass function of $W = X + Y$ and identify its distribution by name. After the calculation, give an intuitive justification for the answer you obtained.

Hints: Figure out what the possible values of W are. To calculate $P(W = a)$ for each possible value a, decompose $P(W = a)$ according to the different values of (X, Y, Z) that make up the event $\{W = a\}$. Use the PMF of (X, Y, Z).

$$P(W=a) = \sum_{i=0}^{a} P(X=i,\ Y=a-i,\ Z=n-a)$$

$$= \sum_{i=0}^{a} \frac{n!}{i!(a-i)!(n-a)!} p^i q^{a-i} r^{n-a}$$

$$= \frac{n!}{(n-a)!} r^{n-a} \cdot \sum_{i=0}^{a} \frac{p^i q^{a-i}}{i!(a-i)!}$$

1.2 b 3 / 3

✓ **- 0 pts** Correct

$B = \{ (x,y) \mid y \geq 3 \}$

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$$= \frac{n!}{(n-a)!} r^{n-a} \cdot \frac{(p+q)^a}{a!}$$

$$= \binom{n}{a} r^{n-a} \cdot (p+q)^a$$

This distribution is $B(n, p+q)$ or $B(n, 1-r)$

We are doing n independent experiments where the outcome is either in Z or not in Z. Binomial distribution can be used to model the number of success (defined as not in Z) in n experiments.

3. Let $n \geq 2$ and $0 < p < 1$. Let $X_1, X_2, \ldots, X_n$ be i.i.d. Bernoulli random variables with success probability p and $S_n = X_1 + \cdots + X_n$. The Bernoulli assumption means that each X_i has PMF $P(X_i = 1) = p = 1 - P(X_i = 0)$.

(a) Compute the conditional joint probability mass function of the random vector $(X_1, \ldots, X_n)$, given that $S_n = k$. That is, find

$$P(X_1 = a_1, X_2 = a_2, \ldots, X_n = a_n \,|\, S_n = k)$$

for all vectors $(a_1, \ldots, a_n)$ of zeros and ones, and all $k \in \{0, \ldots, n\}$.

Hints. Use the definition of conditional probability. Which vectors $(a_1, \ldots, a_n)$ are compatible with $S_n = k$? Check that your conditional PMF sums to 1 over all vectors $(a_1, \ldots, a_n)$.

If $\sum_{i=1}^{n} a_i \neq k$, then

$$P(X_1 = a_1, X_2 = a_2, \cdots, X_n = a_n \,|\, S_n = k) = 0$$

2 2 10 / 10

✓ - 0 pts Correct

$$= \frac{n!}{(n-a)!} r^{n-a} \cdot \sum_{i=0}^{a} \frac{p^i q^{a-i}}{i!(a-i)!}$$

$$= \frac{n!}{(n-a)!} r^{n-a} \cdot \frac{(p+q)^a}{a!}$$

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Otherwise,

$$P(X_1=a_1, X_2=a_2, \cdots, X_n=a_n \mid S_n=k)$$

$$= \frac{P(X_1=a_1, X_2=a_2, \cdots, X_n=a_n, S_n=k)}{P(S_n=k)}$$

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$$= \frac{1/2^n}{\binom{n}{k}(\frac{1}{2})^k(\frac{1}{2})^{n-k}} = \frac{1}{\binom{n}{k}} = \frac{k!(n-k)!}{n!}$$

(b) Deduce whether $X_1, \ldots, X_n$ are conditionally independent, given $S_n = k$. (The general definition is that random variables $X_1, \ldots, X_n$ are conditionally independent, given an event B such that $P(B) > 0$, if

$$P(X_1 = a_1, X_2 = a_2, \ldots, X_n = a_n \mid B) = \prod_{i=1}^{n} P(X_i = a_i \mid B)$$

for all possible values $a_1, \ldots, a_n$.)

Hint. Consider cases $k = 0$, $0 < k < n$ and $k = n$

① For $k=0$,

$$LHS = \begin{cases} 1 & \text{when } a_1=a_2=\cdots=a_n=0 \\ 0 & \text{otherwise} \end{cases}$$

If $a_1=a_2=\cdots=a_n=0$, then

$RHS = 1$, since $P(X_i=0 \mid S_n=0)=1$, $\forall i$

Otherwise

$RHS = 0$, since $\exists i \in \{1,\ldots,n\}$ s.t. $a_i=1$

$\Rightarrow P(X_i=1 \mid S_n=0)=0$

Therefore $LHS = RHS$

3.1 a **5 / 5**

✓ **- 0 pts** Correct

Otherwise,

$$P(X_1 = a_1, X_2 = a_2, \cdots, X_n = a_n \mid S_n = k)$$

$$= \frac{P(X_1 = a_1, X_2 = a_2, \cdots, X_n = a_n, S_n = k)}{P(S_n = k)}$$

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Hint. Consider cases $k = 0$, $0 < k < n$ and $k = n$

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$$LHS = \begin{cases} 1 & \text{when } a_1 = a_2 = \cdots = a_n = 0 \\ 0 & \text{otherwise} \end{cases}$$

If $a_1 = a_2 = \cdots = a_n = 0$, then

$RHS = 1$, since $P(X_i = 0 \mid S_n = 0) = 1$, $\forall i$

Otherwise

$RHS = 0$, since $\exists i \in \{1, \ldots, n\}$ s.t. $a_i = 1$

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Therefore $LHS = RHS$

② For $0 < k < n$,

the equality doesn't hold,

take $a_1 = a_2 = \cdots = a_n = 0$, for example

LHS $= 0$, while RHS $\neq 0$

③ For $k = n$,

$$\text{LHS} = \begin{cases} 1 & \text{when } a_1 = a_2 = \cdots = a_n = 1 \\ 0 & \text{otherwise} \end{cases}$$

If $a_1 = a_2 = \cdots = a_n = 1$, then

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Otherwise,

RHS $= 0$, since $\exists i \in \{1, \cdots, n\}$ s.t. $a_i = 0$

$\Rightarrow P(X_i = 0 \mid S_n = n) = 0$

Therefore LHS $=$ RHS

By ①.②.③, $X_1, \cdots X_n$ are generally NOT conditionally independent, except when $k = 0$ or $k = n$

Further exercises: The following problems from Durrett's book (version *Almost Final Version of the 2nd Edition, December, 2011* on the web): 1.1, 1.2, 1.5, 1.6, 1.7

1.1. A fair coin is tossed repeatedly with results $Y_0, Y_1, Y_2, \ldots$ that are 0 or 1 with probability 1/2 each. For $n \geq 1$ let $X_n = Y_n + Y_{n-1}$ be the number of 1's in the $(n-1)$th and nth tosses. Is X_n a Markov chain?

No, X_n is not a Markov chain

3.2 b **5 / 5**

✓ **- 0 pts** Correct

② For $0<k<n$,

the equality doesn't hold,

take $a_1=a_2=\cdots=a_n=0$, for example

LHS $=0$, while RHS $\neq 0$

③ For $k=n$,

$$\text{LHS} = \begin{cases} 1 & \text{when } a_1=a_2=\cdots=a_n=1 \\ 0 & \text{otherwise} \end{cases}$$

If $a_1=a_2=\cdots=a_n=1$, then

RHS $=1$, since $P(X_i=1|S_n=n)=1$, $\forall i$

Otherwise,

RHS $=0$, since $\exists i\in\{1,\cdots,n\}$ s.t. $a_i=0$

$\Rightarrow P(X_i=0|S_n=n)=0$

Therefore LHS $=$ RHS

By ①.②.③, $X_1,\cdots X_n$ are generally NOT conditionally independent, except when $k=0$ or $k=n$

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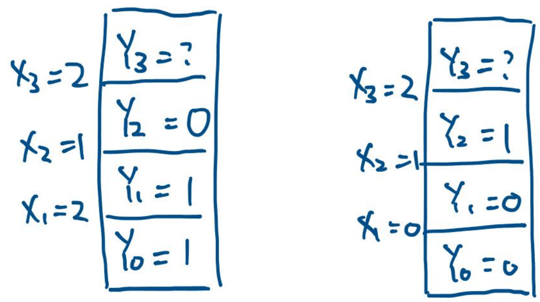
No, X_n is not a Markov chain

$$\mathbb{P}(X_3=2 \mid X_2=1,\ X_1=2) = \mathbb{P}(Y_3=2 \mid Y_2=0,\ Y_1=1,\ Y_0=1) = 0$$

$$\mathbb{P}(X_3=2 \mid X_2=1,\ X_1=0) = \mathbb{P}(Y_3=1 \mid Y_2=1,\ Y_0=0,\ Y_0=0) = \mathbb{P}(Y_3=1) = \tfrac{1}{2}$$



Since the probability of $X_3=2$ depends on the previous two value (i.e. X_2 and X_1), X_n is not a Markov chain.

1.2. Five white balls and five black balls are distributed in two urns in such a way that each urn contains five balls. At each step we draw one ball from each urn and exchange them. Let X_n be the number of white balls in the left urn at time n. Compute the transition probability for X_n.

Let Y_1, Y_2 denote the color of ball from left and right urn respectively.

$$p(1,0) = p(4,5) = \mathbb{P}(Y_1=W,\ Y_2=B) = \tfrac{1}{5}\times\tfrac{1}{5} = 0.04$$

$$p(1,1) = p(4,4) = \mathbb{P}(Y_1=B,\ Y_2=B) + \mathbb{P}(Y_1=W,\ Y_2=W) = \tfrac{1}{5}\times\tfrac{4}{5} + \tfrac{4}{5}\times\tfrac{1}{5} = 0.32$$

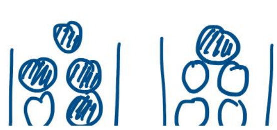
$$p(1,2) = p(4,3) = \mathbb{P}(Y_1=B,\ Y_2=W) = \tfrac{4}{5}\times\tfrac{4}{5} = 0.64$$

$$p(2,1) = p(3,4) = \mathbb{P}(Y_1=W,\ Y_2=B) = \tfrac{2}{5}\times\tfrac{2}{5} = 0.16$$

$$p(2,2) = p(3,3) = \mathbb{P}(Y_1=W,\ Y_2=W) + \mathbb{P}(Y_1=B,\ Y_2=B) = \tfrac{2}{5}\times\tfrac{3}{5} + \tfrac{3}{5}\times\tfrac{2}{5} = 0.48$$

$$p(2,3) = p(3,2) = \mathbb{P}(Y_1=B,\ Y_2=W) = \tfrac{3}{5}\times\tfrac{3}{5} = 0.36$$

$X = 1$



$$\begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 1 & & & & \\ 1 & .04 & .32 & .64 & & & \end{array}$$

4 Textbook 1.1 10 / 10

✓ - 0 pts Correct

$$\mathbb{P}(X_3=2 \mid X_2=1,\ X_1=2) = \mathbb{P}(Y_3=2 \mid Y_2=0,\ Y_1=1,\ Y_0=1) = 0$$
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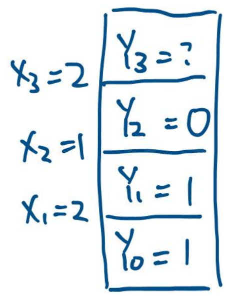

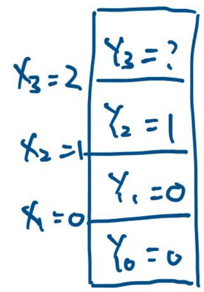

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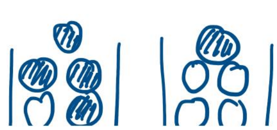
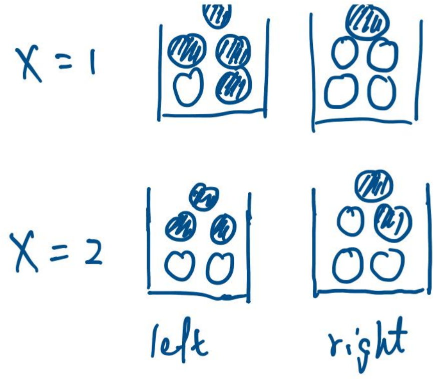
Let Y_1, Y_2 denote the color of ball from left and right urn respectively.

$$p(1,0) = p(4,5) = \mathbb{P}(Y_1=W,\ Y_2=B) = \frac{1}{5}\times\frac{1}{5} = 0.04$$
$$p(1,1) = p(4,4) = \mathbb{P}(Y_1=B,\ Y_2=B) + \mathbb{P}(Y_1=W,\ Y_2=W) = \frac{1}{5}\times\frac{4}{5} + \frac{4}{5}\times\frac{1}{5} = 0.32$$
$$p(1,2) = p(4,3) = \mathbb{P}(Y_1=B,\ Y_2=W) = \frac{4}{5}\times\frac{4}{5} = 0.64$$
$$p(2,1) = p(3,4) = \mathbb{P}(Y_1=W,\ Y_2=B) = \frac{2}{5}\times\frac{2}{5} = 0.16$$
$$p(2,2) = p(3,3) = \mathbb{P}(Y_1=W,\ Y_2=W) + \mathbb{P}(Y_1=B,\ Y_2=B) = \frac{2}{5}\times\frac{3}{5} + \frac{3}{5}\times\frac{2}{5} = 0.48$$
$$p(2,3) = p(3,2) = \mathbb{P}(Y_1=B,\ Y_2=W) = \frac{3}{5}\times\frac{3}{5} = 0.36$$

$X=1$ 

$$\begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 1 & & & & \\ 1 & .04 & .32 & .64 & & & \end{array}$$



$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{bmatrix} 0 & 1 & & & & \\ .04 & .32 & .64 & & & \\ & .16 & .48 & .36 & & \\ & & .36 & .48 & .16 & \\ & & & .64 & .32 & 0.04 \\ & & & & 1 & 0 \end{bmatrix}$$

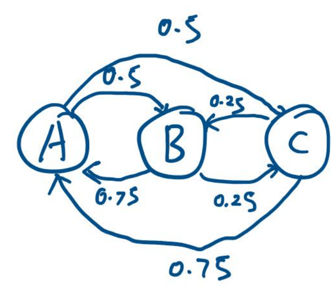
1.5. Consider a gambler's ruin chain with $N = 4$. That is, if $1 \leq i \leq 3$, $p(i, i+1) = 0.4$, and $p(i, i-1) = 0.6$, but the endpoints are absorbing states: $p(0,0) = 1$ and $p(4,4) = 1$ Compute $p^3(1,4)$ and $p^3(1,0)$.

$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & & & \\ 1 & .6 & 0 & .4 & & \\ 2 & & .6 & 0 & .4 & \\ 3 & & & .6 & 0 & .4 \\ 4 & & & & 0 & 1 \end{array} \Rightarrow P^3 = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & .744 & 0 & .192 & 0 & .064 \\ 2 & .36 & .288 & 0 & .192 & .16 \\ 3 & .216 & 0 & .288 & 0 & .496 \\ 4 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$\Rightarrow p^3(1,4) = 0.064 \qquad p^3(1,0) = 0.744$

1.6. A taxicab driver moves between the airport A and two hotels B and C according to the following rules. If he is at the airport, he will be at one of the two hotels next with equal probability. If at a hotel then he returns to the airport with probability 3/4 and goes to the other hotel with probability 1/4. (a) Find the transition matrix for the chain. (b) Suppose the driver begins at the airport at time 0. Find the probability for each of his three possible locations at time 2 and the probability he is at hotel B at time 3.

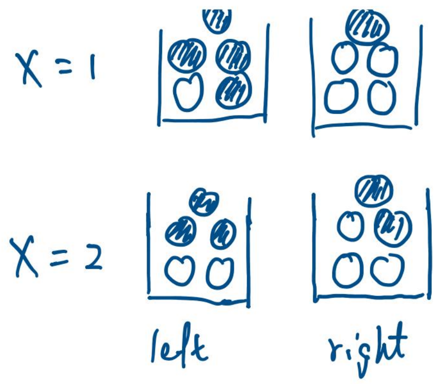
(a) $P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & .5 & .5 \\ B & .75 & 0 & .25 \\ C & .75 & .25 & 0 \end{array}$



$\begin{array}{c|ccc} & A & B & C \\ \hline A & .75 & 125 & 125 \end{array}$ probability for each location at T=2

5 Textbook 1.2 **10 / 10**

✓ **- 0 pts** **Correct**



$$P = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{bmatrix} 0 & 1 & & & & \\ .04 & .32 & .64 & & & \\ & .16 & .48 & .36 & & \\ & & .36 & .48 & .16 & \\ & & & .64 & .32 & 0.04 \\ & & & & 1 & 0 \end{bmatrix}$$

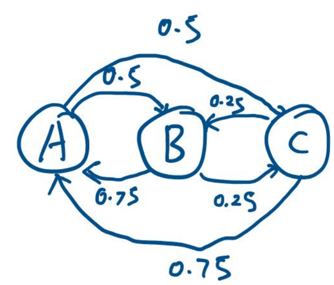
1.5. Consider a gambler's ruin chain with $N = 4$. That is, if $1 \le i \le 3$, $p(i, i+1) = 0.4$, and $p(i, i-1) = 0.6$, but the endpoints are absorbing states: $p(0,0) = 1$ and $p(4,4) = 1$ Compute $p^3(1,4)$ and $p^3(1,0)$.

$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & & & \\ 1 & .6 & 0 & .4 & & \\ 2 & & .6 & 0 & .4 & \\ 3 & & & .6 & 0 & .4 \\ 4 & & & & 0 & 1 \end{array} \Rightarrow P^3 = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & .744 & 0 & .192 & 0 & .064 \\ 2 & .36 & .288 & 0 & .192 & .16 \\ 3 & .216 & 0 & .288 & 0 & .496 \\ 4 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$$\Rightarrow p^3(1,4) = 0.064 \qquad p^3(1,0) = 0.744$$

1.6. A taxicab driver moves between the airport A and two hotels B and C according to the following rules. If he is at the airport, he will be at one of the two hotels next with equal probability. If at a hotel then he returns to the airport with probability 3/4 and goes to the other hotel with probability 1/4. (a) Find the transition matrix for the chain. (b) Suppose the driver begins at the airport at time 0. Find the probability for each of his three possible locations at time 2 and the probability he is at hotel B at time 3.

(a) $$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & .5 & .5 \\ B & .75 & 0 & .25 \\ C & .75 & .25 & 0 \end{array}$$

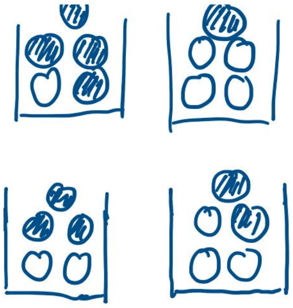


$$\begin{array}{c|ccc} & A & B & C \\ \hline A & .75 & .125 & .125 \end{array}$$ probability for each location at T=2

6 Textbook 1.5 10 / 10

✓ **- 0 pts** **Correct**

$X = 1$



$X = 2$

left right

$$\mathcal{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{bmatrix} 0 & 1 & & & & \\ .04 & .32 & .64 & & & \\ & .16 & .48 & .36 & & \\ & & .36 & .48 & .16 & \\ & & & .64 & .32 & 0.04 \\ & & & & 1 & 0 \end{bmatrix}$$

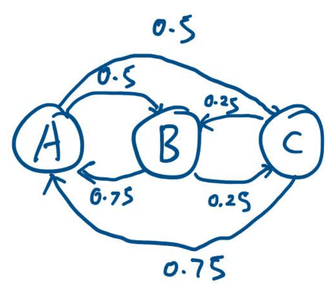
1.5. Consider a gambler's ruin chain with $N = 4$. That is, if $1 \le i \le 3$, $p(i, i+1) = 0.4$, and $p(i, i-1) = 0.6$, but the endpoints are absorbing states: $p(0,0) = 1$ and $p(4,4) = 1$ Compute $p^3(1,4)$ and $p^3(1,0)$.

$$\mathcal{P} = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & & & \\ 1 & .6 & 0 & .4 & & \\ 2 & & .6 & 0 & .4 & \\ 3 & & & .6 & 0 & .4 \\ 4 & & & & 0 & 1 \end{array} \Rightarrow \mathcal{P}^3 = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & .744 & 0 & .192 & 0 & .064 \\ 2 & .36 & .288 & 0 & .192 & .16 \\ 3 & .216 & 0 & .288 & 0 & .496 \\ 4 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$\Rightarrow p^3(1,4) = 0.064 \qquad p^3(1,0) = 0.744$

1.6. A taxicab driver moves between the airport A and two hotels B and C according to the following rules. If he is at the airport, he will be at one of the two hotels next with equal probability. If at a hotel then he returns to the airport with probability 3/4 and goes to the other hotel with probability 1/4. (a) Find the transition matrix for the chain. (b) Suppose the driver begins at the airport at time 0. Find the probability for each of his three possible locations at time 2 and the probability he is at hotel B at time 3.

(a) $$\mathcal{P} = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & .5 & .5 \\ B & .75 & 0 & .25 \\ C & .75 & .25 & 0 \end{array}$$



$$\begin{array}{c|ccc} & A & B & C \\ \hline A & .75 & 125 & 125 \end{array}$$ probability for each location at T=2

7.1 a **4 / 4**

✓ **- 0 pts** Correct

(b) $$\mathcal{P}^2 = \begin{array}{c|ccc} & A & B & C \\ \hline A & .75 & .125 & .125 \\ B & .1875 & .4375 & .375 \\ C & .1875 & .375 & .4375 \end{array}$$

probability for each location at $T=2$

$A = 0.75$
$B = 0.125$
$C = 0.125$

$$\mathcal{P}^3 = \begin{array}{c|ccc} & A & B & C \\ \hline A & .1875 & .40625 & .40625 \\ B & .60938 & .1875 & .20313 \\ C & .60938 & .20313 & .1875 \end{array}$$

probability he is at hotel B at time 3 is 0.40625

1.7. Suppose that the probability it rains today is 0.3 if neither of the last two days was rainy, but 0.6 if at least one of the last two days was rainy. Let the weather on day n, W_n, be R for rain, or S for sun. W_n is not a Markov chain, but the weather for the last two days $X_n = (W_{n-1}, W_n)$ is a Markov chain with four states $\{RR, RS, SR, SS\}$. (a) Compute its transition probability. (b) Compute the two-step transition probability. (c) What is the probability it will rain on Wednesday given that it did not rain on Sunday or Monday.

(a) $\mathbb{P}(R \mid SS) = 0.3$

$\Rightarrow \mathbb{P}(S \mid SS) = 0.7$

$\mathbb{P}(R \mid RR) = \mathbb{P}(R \mid SR) = \mathbb{P}(R \mid RS) = 0.6$

$\Rightarrow \mathbb{P}(S \mid RR) = \mathbb{P}(S \mid SR) = \mathbb{P}(S \mid RS) = 0.4$

* The first, second, and third letter represents W_n, W_{n-1}, W_{n-2} respectively

$$\mathcal{P} = \begin{array}{c|cccc} & RR & RS & SR & SS \\ \hline RR & 0.6 & 0.4 & & \\ RS & & & 0.6 & 0.4 \\ SR & 0.6 & 0.4 & & \\ SS & & & 0.3 & 0.7 \end{array}$$

(b) $$\mathcal{P}^2 = \begin{array}{c|cccc} & RR & RS & SR & SS \\ \hline RR & .36 & .24 & .24 & .16 \\ RS & .36 & .24 & .12 & .28 \\ SR & .36 & .24 & .24 & .16 \\ SS & .18 & .12 & .21 & .49 \end{array}$$

(c) $\boxed{S}\ \boxed{S}\ \boxed{?}\ \boxed{R}$

S M T W

$p^2(SS, SR) + p^2(SS, RR)$

$= 0.21 + 0.18 = 0.39$

7.2 b **6 / 6**

✓ **- 0 pts** Correct

(b) $\mathcal{P}^2 = \begin{array}{c|ccc} & A & B & C \\ \hline A & .75 & .125 & .125 \\ B & .1875 & .4375 & .375 \\ C & .1875 & .375 & .4375 \end{array}$ probability for each location at $T=2$

$A = 0.75$
$B = 0.125$
$C = 0.125$

$\mathcal{P}^3 = \begin{array}{c|ccc} & A & B & C \\ \hline A & .1875 & .40625 & .40625 \\ B & .60938 & .1875 & .20313 \\ C & .60938 & .20313 & .1875 \end{array}$ probability he is at hotel B at time 3 is 0.40625

1.7. Suppose that the probability it rains today is 0.3 if neither of the last two days was rainy, but 0.6 if at least one of the last two days was rainy. Let the weather on day n, W_n, be R for rain, or S for sun. W_n is not a Markov chain, but the weather for the last two days $X_n = (W_{n-1}, W_n)$ is a Markov chain with four states $\{RR, RS, SR, SS\}$. (a) Compute its transition probability. (b) Compute the two-step transition probability. (c) What is the probability it will rain on Wednesday given that it did not rain on Sunday or Monday.

(a) $\mathbb{P}(R|SS) = 0.3$
$\Rightarrow \mathbb{P}(S|SS) = 0.7$
$\mathbb{P}(R|RR) = \mathbb{P}(R|SR) = \mathbb{P}(R|RS) = 0.6$
$\Rightarrow \mathbb{P}(S|RR) = \mathbb{P}(S|SR) = \mathbb{P}(S|RS) = 0.4$
* The first, second, and third letter represents W_n, W_{n-1}, W_{n-2} respectively

$$\mathcal{P} = \begin{array}{c|cccc} & RR & RS & SR & SS \\ \hline RR & 0.6 & 0.4 & & \\ RS & & & 0.6 & 0.4 \\ SR & 0.6 & 0.4 & & \\ SS & & & 0.3 & 0.7 \end{array}$$

(b) $$\mathcal{P}^2 = \begin{array}{c|cccc} & RR & RS & SR & SS \\ \hline RR & .36 & .24 & .24 & .16 \\ RS & .36 & .24 & .12 & .28 \\ SR & .36 & .24 & .24 & .16 \\ SS & .18 & .12 & .21 & .49 \end{array}$$

(c) $\boxed{S}\boxed{S}\boxed{?}\boxed{R}$
S M T W

$p^2(SS, SR) + p^2(SS, RR)$
$= 0.21 + 0.18 = 0.39$

8.1 a **4 / 4**

✓ **- 0 pts** Correct

(b) $$\mathcal{P}^2 = \begin{array}{c} \\ A \\ B \\ C \end{array}\!\!\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} .75 & .125 & .125 \\ .1875 & .4375 & .375 \\ .1875 & .375 & .4375 \end{bmatrix} \end{array}$$

probability for each location at $T=2$

$A = 0.75$
$B = 0.125$
$C = 0.125$

$$\mathcal{P}^3 = \begin{array}{c} \\ A \\ B \\ C \end{array}\!\!\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} .1875 & .40625 & .40625 \\ .60938 & .1875 & .20313 \\ .60938 & .20313 & .1875 \end{bmatrix} \end{array}$$

probability he is at hotel B at time 3 is 0.40625

1.7. Suppose that the probability it rains today is 0.3 if neither of the last two days was rainy, but 0.6 if at least one of the last two days was rainy. Let the weather on day n, W_n, be R for rain, or S for sun. W_n is not a Markov chain, but the weather for the last two days $X_n = (W_{n-1}, W_n)$ is a Markov chain with four states $\{RR, RS, SR, SS\}$. (a) Compute its transition probability. (b) Compute the two-step transition probability. (c) What is the probability it will rain on Wednesday given that it did not rain on Sunday or Monday.

(a) $\mathbb{P}(R|SS) = 0.3$

$\Rightarrow \mathbb{P}(S|SS) = 0.7$

$\mathbb{P}(R|RR) = \mathbb{P}(R|SR) = \mathbb{P}(R|RS) = 0.6$

$\Rightarrow \mathbb{P}(S|RR) = \mathbb{P}(S|SR) = \mathbb{P}(S|RS) = 0.4$

* The first, second, and third letter represents W_n, W_{n-1}, W_{n-2} respectively

$$\mathcal{P} = \begin{array}{c} \\ RR \\ RS \\ SR \\ SS \end{array}\!\!\begin{array}{c} \begin{array}{cccc} RR & RS & SR & SS \end{array} \\ \begin{bmatrix} 0.6 & 0.4 & & \\ & & 0.6 & 0.4 \\ 0.6 & 0.4 & & \\ & & 0.3 & 0.7 \end{bmatrix} \end{array}$$

(b) $$\mathcal{P}^2 = \begin{array}{c} \\ RR \\ RS \\ SR \\ SS \end{array}\!\!\begin{array}{c} \begin{array}{cccc} RR & RS & SR & SS \end{array} \\ \begin{bmatrix} .36 & .24 & .24 & .16 \\ .36 & .24 & .12 & .28 \\ .36 & .24 & .24 & .16 \\ .18 & .12 & .21 & .49 \end{bmatrix} \end{array}$$

(c) $\boxed{S}\boxed{S}\boxed{?}\boxed{R}$

S M T W

$p^2(SS, SR) + p^2(SS, RR)$

$= 0.21 + 0.18 = 0.39$

8.2 b **3 / 3**

✓ **- 0 pts** Correct

(b) $\mathcal{P}^2 = \begin{array}{c|ccc} & A & B & C \\ \hline A & .75 & .125 & .125 \\ B & .1875 & .4375 & .375 \\ C & .1875 & .375 & .4375 \end{array}$

probability for each location at $T=2$

$A = 0.75$
$B = 0.125$
$C = 0.125$

$\mathcal{P}^3 = \begin{array}{c|ccc} & A & B & C \\ \hline A & .1875 & .40625 & .40625 \\ B & .60938 & .1875 & .20313 \\ C & .60938 & .20313 & .1875 \end{array}$

probability he is at hotel B at time 3 is 0.40625

1.7. Suppose that the probability it rains today is 0.3 if neither of the last two days was rainy, but 0.6 if at least one of the last two days was rainy. Let the weather on day n, W_n, be R for rain, or S for sun. W_n is not a Markov chain, but the weather for the last two days $X_n = (W_{n-1}, W_n)$ is a Markov chain with four states $\{RR, RS, SR, SS\}$. (a) Compute its transition probability. (b) Compute the two-step transition probability. (c) What is the probability it will rain on Wednesday given that it did not rain on Sunday or Monday.

(a) $\mathbb{P}(R|SS) = 0.3$

$\Rightarrow \mathbb{P}(S|SS) = 0.7$

$\mathbb{P}(R|RR) = \mathbb{P}(R|SR) = \mathbb{P}(R|RS) = 0.6$

$\Rightarrow \mathbb{P}(S|RR) = \mathbb{P}(S|SR) = \mathbb{P}(S|RS) = 0.4$

* The first, second, and third letter represents W_n, W_{n-1}, W_{n-2} respectively

$\mathcal{P} = \begin{array}{c|cccc} & RR & RS & SR & SS \\ \hline RR & 0.6 & 0.4 & & \\ RS & & & 0.6 & 0.4 \\ SR & 0.6 & 0.4 & & \\ SS & & & 0.3 & 0.7 \end{array}$

(b) $\mathcal{P}^2 = \begin{array}{c|cccc} & RR & RS & SR & SS \\ \hline RR & .36 & .24 & .24 & .16 \\ RS & .36 & .24 & .12 & .28 \\ SR & .36 & .24 & .24 & .16 \\ SS & .18 & .12 & .21 & .49 \end{array}$

(c) $\boxed{S}\boxed{S}\boxed{?}\boxed{R}$

S M T W

$p^2(SS, SR) + p^2(SS, RR)$

$= 0.21 + 0.18 = 0.39$

8.3 C **3 / 3**

✓ **- 0 pts** Correct