Math 632 004 HW02

Shawn Zhong

TOTAL POINTS

69 / 72

QUESTION 1

1 Problem 1 **10 / 10**

✓ **- 0 pts** Correct

QUESTION 2

2 Problem 2 **10 / 10**

✓ **- 0 pts** Correct

QUESTION 3

Problem 3 10 pts

3.1 3(a) **3 / 3**

✓ **- 0 pts** Correct

3.2 3(b) **3 / 3**

✓ **- 0 pts** Correct

3.3 3(c) **2 / 2**

✓ **- 0 pts** Correct

3.4 3(d) **2 / 2**

✓ **- 0 pts** Correct

QUESTION 4

Problem 4 14 pts

4.1 4(a) **2 / 2**

✓ **- 0 pts** Correct

4.2 4(b) **2 / 2**

✓ **- 0 pts** Correct

4.3 4(c) **2 / 2**

✓ **- 0 pts** Correct

4.4 4(d) **2 / 2**

✓ **- 0 pts** Correct

4.5 4(e) **2 / 2**

✓ **- 0 pts** Correct

4.6 4(f) **2 / 2**

✓ **- 0 pts** Correct

4.7 4(g) **2 / 2**

✓ **- 0 pts** Correct

QUESTION 5

Problem 5: Durrett 1.8 8 pts

5.1 1.8(a) **2 / 2**

✓ **- 0 pts** Correct

5.2 1.8(b) **2 / 2**

✓ **- 0 pts** Correct

5.3 1.8(c) **2 / 2**

✓ **- 0 pts** Correct

5.4 1.8(d) **2 / 2**

✓ **- 0 pts** Correct

QUESTION 6

Problem 6 10 pts

6.1 6(a) **4 / 4**

✓ **- 0 pts** Correct

6.2 6(b) **2 / 3**

✓ **- 1 pts** The only irreducible sets are singletons

6.3 6(c) **3 / 3**

✓ - 0 pts Correct

QUESTION 7

Problem 7 10 pts

7.1 7(a) 3 / 5

✓ - 2 pts The definition of conditional probability is not correctly used

7.2 7(b) 5 / 5

✓ - 0 pts Correct

HW2 - Problem & Answer

Tuesday, September 18, 2018 7:59 PM

Math 632 Lecture 4, Fall 2018, Homework 2

Due Thursday, September 20, 9:30am

This homework set has 7 problems: 1 problem is from Durrett's book, and there are 6 problems to supplement that material.

Please note that the expression "Durrett's book" refers to the version *Almost Final Version of the 2nd Edition, December, 2011.*

Please check the homework instructions on the course homepage. In particular:

- Credit comes from your reasoning, not your numerical answer.
- Observe rules of academic integrity. You are encouraged to discuss the problems with fellow students, but copied work is not acceptable and will result in zero credit.
- You may find solutions to some exercises on the web. However, the point of the homework is to give you the problem solving practice you need in the exams. Hence it is not smart to take shortcuts to secure the few points that come from homework.

Question 1

1. Consider a discrete time Markov chain with transition probabilities $p(i,j)$, with state space $\{1,2,\ldots,10\}$, and assume $X_0 = 3$. Express

$$P(X_6 = 7, X_5 = 3 | X_4 = 1, X_9 = 3, X_0 = 3)$$

in terms of the (if necessary multi-step) transition probabilities.

- $\mathbb{P}(X_6 = 7, X_5 = 3 | X_4 = 1, X_9 = 3, X_0 = 3)$
- $= \mathbb{P}(X_6 = 7, X_5 = 3 | X_4 = 1, X_9 = 3)$, by Markov property
- $= \dfrac{\mathbb{P}(X_6 = 7, X_5 = 3, X_4 = 1, X_9 = 3)}{\mathbb{P}(X_4 = 1, X_9 = 3)}$, by Bayes' law
- $= \dfrac{p(1,3)p(3,7)p^3(7,3)}{p^5(1,3)}$, assume $p^5(1,3) \neq 0$

Question 2

2. Consider a discrete time Markov chain with transition probabilities $p(i,j)$, with state space $\{1,2,\ldots,10\}$. Assume, moreover, that T is a stopping time with the properties $P_1(T < \infty) = 1$ and $P(X_T = 3) = 1$. Express

$$P(X_{T+6} = 7, X_{T+5} = 3 | X_{T+4} = 1, X_{T+9} = 3, X_0 = 1)$$

in terms of the (if necessary multi-step) transition probabilities.

1 Problem 1 **10 / 10**

✓ **- 0 pts** **Correct**

HW2 - Problem & Answer

Tuesday, September 18, 2018 7:59 PM

Math 632 Lecture 4, Fall 2018, Homework 2

Due Thursday, September 20, 9:30am

This homework set has 7 problems: 1 problem is from Durrett's book, and there are 6 problems to supplement that material.

Please note that the expression "Durrett's book" refers to the version *Almost Final Version of the 2nd Edition, December, 2011.*

Please check the homework instructions on the course homepage. In particular:

- Credit comes from your reasoning, not your numerical answer.
- Observe rules of academic integrity. You are encouraged to discuss the problems with fellow students, but copied work is not acceptable and will result in zero credit.
- You may find solutions to some exercises on the web. However, the point of the homework is to give you the problem solving practice you need in the exams. Hence it is not smart to take shortcuts to secure the few points that come from homework.

Question 1

1. Consider a discrete time Markov chain with transition probabilities $p(i,j)$, with state space $\{1,2,\ldots,10\}$, and assume $X_0=3$. Express

$$P(X_6=7,X_5=3|X_4=1,X_9=3,X_0=3)$$

in terms of the (if necessary multi-step) transition probabilities.

- $\mathbb{P}(X_6=7,X_5=3|X_4=1,X_9=3,X_0=3)$
- $=\mathbb{P}(X_6=7,X_5=3|X_4=1,X_9=3)$, by Markov property
- $=\dfrac{\mathbb{P}(X_6=7,X_5=3,X_4=1,X_9=3)}{\mathbb{P}(X_4=1,X_9=3)}$, by Bayes' law
- $=\dfrac{p(1,3)p(3,7)p^3(7,3)}{p^5(1,3)}$, assume $p^5(1,3)\neq 0$

Question 2

2. Consider a discrete time Markov chain with transition probabilities $p(i,j)$, with state space $\{1,2,\ldots,10\}$. Assume, moreover, that T is a stopping time with the properties $P_1(T<\infty)=1$ and $P(X_T=3)=1$. Express

$$P(X_{T+6}=7,X_{T+5}=3|X_{T+4}=1,X_{T+9}=3,X_0=1)$$

in terms of the (if necessary multi-step) transition probabilities.

- $\begin{cases} \mathbb{P}_1(T < \infty) = 1 \\ \mathbb{P}(X_T = 3) = 3 \end{cases} \Rightarrow p^n(1,3) \neq 0, \forall n \in \{1,2,3,\dots\}$
- $\mathbb{P}(X_{T+6} = 7, X_{T+5} = 3 | X_{T+4} = 1, X_{T+9} = 3, X_0 = 3)$
- $= \mathbb{P}(X_6 = 7, X_5 = 3 | X_4 = 1, X_9 = 3)$, by strong Markov property
- $= \dfrac{\mathbb{P}(X_6 = 7, X_5 = 3, X_4 = 1, X_9 = 3)}{\mathbb{P}(X_4 = 1, X_9 = 3)}$, by Bayes' law
- $= \dfrac{p(1,3)p(3,7)p^3(7,3)}{p^5(1,3)}$

Question 3

3. Consider the discrete time Markov chain with state space $\{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0.3 & 0.2 & 0.1 & 0.4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix}$$

(a) Identify the transient and the recurrent states.

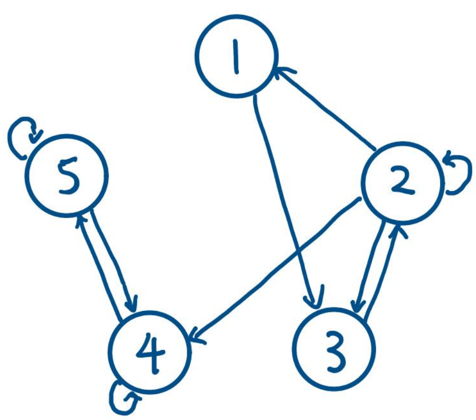
(b) Let $R_2 = \max\{n \geq 0 : X_n = 2\}$. Prove that $P_2(R_2 < \infty) = 1$.

(c) Is R_2 a stopping time? Argue why or why not.

(d) Find $P_3(X_{R_2+1} = 4 | X_{R_2} = 2, R_2 = 8)$.

Part (a)

- Transient: 1, 2, 3
- Recurrent: 4, 5



Part (b)

- $\rho_{22} \leq 1 - p(2, 4) < 1$
- $\Rightarrow \rho_{22}^k \to 0$ as $k \to \infty$
- $\Rightarrow \mathbb{P}_2(R_2 = \infty) = 0$
- $\Rightarrow \mathbb{P}_2(R_2 < \infty) = 1$

Part (c)

2 Problem 2 **10 / 10**

✓ **- 0 pts** Correct

- $\begin{cases} \mathbb{P}_1(T < \infty) = 1 \\ \mathbb{P}(X_T = 3) = 3 \end{cases} \Rightarrow p^n(1,3) \neq 0, \forall n \in \{1,2,3,\ldots\}$
- $\mathbb{P}(X_{T+6} = 7, X_{T+5} = 3 | X_{T+4} = 1, X_{T+9} = 3, X_0 = 3)$
- $= \mathbb{P}(X_6 = 7, X_5 = 3 | X_4 = 1, X_9 = 3)$, by strong Markov property
- $= \dfrac{\mathbb{P}(X_6 = 7, X_5 = 3, X_4 = 1, X_9 = 3)}{\mathbb{P}(X_4 = 1, X_9 = 3)}$, by Bayes' law
- $= \dfrac{p(1,3)p(3,7)p^3(7,3)}{p^5(1,3)}$

Question 3

3. Consider the discrete time Markov chain with state space $\{1,2,3,4,5\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0.3 & 0.2 & 0.1 & 0.4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix}$$

(a) Identify the transient and the recurrent states.

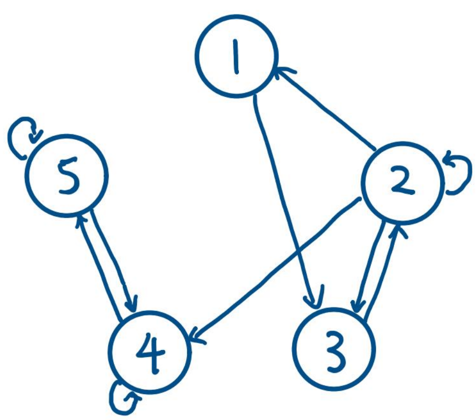
(b) Let $R_2 = \max\{n \geq 0 : X_n = 2\}$. Prove that $P_2(R_2 < \infty) = 1$.

(c) Is R_2 a stopping time? Argue why or why not.

(d) Find $P_3(X_{R_2+1} = 4 | X_{R_2} = 2, R_2 = 8)$.

Part (a)

- Transient: 1, 2, 3
- Recurrent: 4, 5



Part (b)

- $\rho_{22} \leq 1 - p(2,4) < 1$
- $\Rightarrow \rho_{22}^k \to 0$ as $k \to \infty$
- $\Rightarrow \mathbb{P}_2(R_2 = \infty) = 0$
- $\Rightarrow \mathbb{P}_2(R_2 < \infty) = 1$

Part (c)

3.1 3(a) **3 / 3**

✓ **- 0 pts** Correct

- $\begin{cases} \mathbb{P}_1(T < \infty) = 1 \\ \mathbb{P}(X_T = 3) = 3 \end{cases} \Rightarrow p^n(1,3) \neq 0, \forall n \in \{1,2,3,\dots\}$
- $\mathbb{P}\big(X_{T+6} = 7, X_{T+5} = 3 | X_{T+4} = 1, X_{T+9} = 3, X_0 = 3\big)$
- $= \mathbb{P}\big(X_6 = 7, X_5 = 3 | X_4 = 1, X_9 = 3\big)$, by strong Markov property
- $= \dfrac{\mathbb{P}\big(X_6 = 7, X_5 = 3, X_4 = 1, X_9 = 3\big)}{\mathbb{P}\big(X_4 = 1, X_9 = 3\big)}$, by Bayes' law
- $= \dfrac{p(1,3)p(3,7)p^3(7,3)}{p^5(1,3)}$

Question 3

3. Consider the discrete time Markov chain with state space $\{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0.3 & 0.2 & 0.1 & 0.4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix}$$

(a) Identify the transient and the recurrent states.

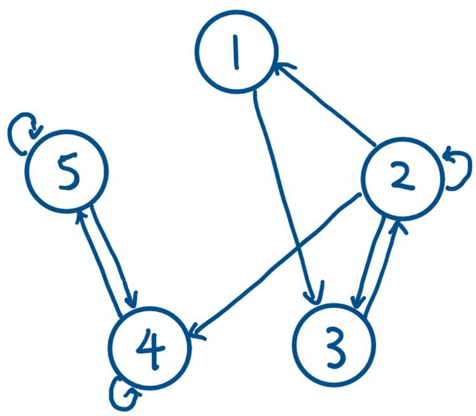
(b) Let $R_2 = \max\{n \geq 0 : X_n = 2\}$. Prove that $P_2(R_2 < \infty) = 1$.

(c) Is R_2 a stopping time? Argue why or why not.

(d) Find $P_3(X_{R_2+1} = 4 | X_{R_2} = 2, R_2 = 8)$.

Part (a)

- Transient: 1, 2, 3
- Recurrent: 4, 5



Part (b)

- $\rho_{22} \leq 1 - p(2, 4) < 1$
- $\Rightarrow \rho_{22}^k \to 0$ as $k \to \infty$
- $\Rightarrow \mathbb{P}_2\big(R_2 = \infty\big) = 0$
- $\Rightarrow \mathbb{P}_2\big(R_2 < \infty\big) = 1$

Part (c)

3.2 3(b) 3 / 3

✓ - 0 pts Correct

- $\begin{cases} \mathbb{P}_1(T < \infty) = 1 \\ \mathbb{P}(X_T = 3) = 3 \end{cases} \Rightarrow p^n(1,3) \neq 0, \forall n \in \{1,2,3,\dots\}$
- $\mathbb{P}(X_{T+6} = 7, X_{T+5} = 3 | X_{T+4} = 1, X_{T+9} = 3, X_0 = 3)$
- $= \mathbb{P}(X_6 = 7, X_5 = 3 | X_4 = 1, X_9 = 3)$, by strong Markov property
- $= \dfrac{\mathbb{P}(X_6 = 7, X_5 = 3, X_4 = 1, X_9 = 3)}{\mathbb{P}(X_4 = 1, X_9 = 3)}$, by Bayes' law
- $= \dfrac{p(1,3)p(3,7)p^3(7,3)}{p^5(1,3)}$

Question 3

3. Consider the discrete time Markov chain with state space $\{1,2,3,4,5\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0.3 & 0.2 & 0.1 & 0.4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix}$$

(a) Identify the transient and the recurrent states.

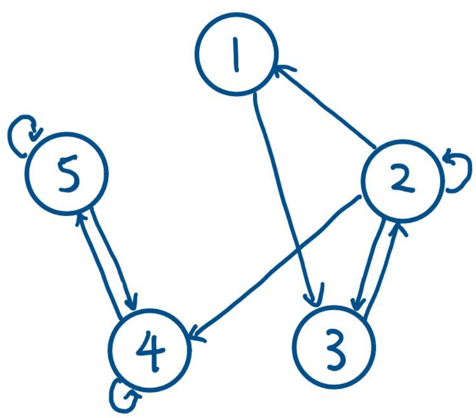
(b) Let $R_2 = \max\{n \geq 0 : X_n = 2\}$. Prove that $P_2(R_2 < \infty) = 1$.

(c) Is R_2 a stopping time? Argue why or why not.

(d) Find $P_3(X_{R_2+1} = 4 | X_{R_2} = 2, R_2 = 8)$.

Part (a)

- Transient: 1, 2, 3
- Recurrent: 4, 5



Part (b)

- $\rho_{22} \leq 1 - p(2,4) < 1$
- $\Rightarrow \rho_{22}^k \to 0$ as $k \to \infty$
- $\Rightarrow \mathbb{P}_2(R_2 = \infty) = 0$
- $\Rightarrow \mathbb{P}_2(R_2 < \infty) = 1$

Part (c)

- No, because $\{R_2 = n\} = \{X_n = 2, X_{n+1} \neq 2, X_{n+2} \neq 2, \ldots\}$ depends on $X_{n+1}, X_{n+2}, \ldots$

Part (d)

- Since $X_{R_2} = 2$ and $R_2 = 8$, the last visit to state 2 is X_8
- If we go to state 1 or 2, we must go back to state 2 again
- Therefore, the next state must be 4
- i.e. $\mathbb{P}_3\left(X_{R_2+1} = 4 \middle| X_{R_2} = 2, R_2 = 8\right) = 1$

Question 4

4. Consider a Markov chain with state space $S = \{1, 2\}$, with transition matrix

$$P = \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}.$$

Decide which of the following is a stopping time:

(a) $T_1 = \min\{n \geq 6 : X_n = 2\}$
(b) $T_2 = \min\{n \geq 1 : X_{n+1} = 2\}$
(c) $T_3 = \min\{n \geq 2 : X_{n-1} = 2\}$
(d) $T_4 = \min\{n > T_1 : X_{n-1} = 2\}$
(e) $T_5 = \min\{n \geq 10 : X_n = X_{n-1}\}$
(f) $T_6 = \min\{n \geq 1 : X_n = X_5\}$
(g) $T_7 = 10$

(a) T_1 is a stopping time
  - $\{T_1 = n\} = \{X_n = 2, X_{n-1} \neq 2, \ldots, X_6 \neq 2\}$

(b) T_2 is **NOT** a stopping time
  - $\{T_2 = n\} = \{X_{n+1} = 2, X_n \neq 2, \ldots, X_1 \neq 2\}$ dependes on X_{n+1}

(c) T_3 is a stopping time
  - $\{T_3 = n\} = \{X_{n-1} = 2, X_{n-2} \neq 2, \ldots, X_1 \neq 2\}$

(d) T_4 is a stopping time
  - $\{T_4 = n\} = \{T_1 = n - 1\} = \{X_{n-1} = 2, X_{n-2} \neq 2, \ldots, X_6 \neq 2\}$

(e) T_5 is a stopping time
  - $\{T_5 = n\} = \{X_n = X_{n-1}, X_{n-1} \neq X_{n-2}, \ldots, X_{10} \neq X_9\}$

(f) T_6 is **NOT** a stopping time
  - $\{T_6 = 1\} = \{X_1 = X_5\}$ dependes on X_5

(g) T_7 is a stopping time
  - Since $\{T_7 = n\}$ does not depends on $X_i, \forall i \in \mathbb{N}$

Question 5

3.3 3(c) **2 / 2**

✓ **- 0 pts** Correct

- No, because $\{R_2 = n\} = \{X_n = 2, X_{n+1} \neq 2, X_{n+2} \neq 2, \ldots\}$ depends on $X_{n+1}, X_{n+2}, \ldots$

Part (d)

- Since $X_{R_2} = 2$ and $R_2 = 8$, the last visit to state 2 is X_8
- If we go to state 1 or 2, we must go back to state 2 again
- Therefore, the next state must be 4
- i.e. $\mathbb{P}_3\left(X_{R_2+1} = 4 \middle| X_{R_2} = 2, R_2 = 8\right) = 1$

Question 4

4. Consider a Markov chain with state space $S = \{1, 2\}$, with transition matrix

$$P = \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}.$$

Decide which of the following is a stopping time:

(a) $T_1 = \min\{n \geq 6 : X_n = 2\}$
(b) $T_2 = \min\{n \geq 1 : X_{n+1} = 2\}$
(c) $T_3 = \min\{n \geq 2 : X_{n-1} = 2\}$
(d) $T_4 = \min\{n > T_1 : X_{n-1} = 2\}$
(e) $T_5 = \min\{n \geq 10 : X_n = X_{n-1}\}$
(f) $T_6 = \min\{n \geq 1 : X_n = X_5\}$
(g) $T_7 = 10$

(a) T_1 is a stopping time
   - $\{T_1 = n\} = \{X_n = 2, X_{n-1} \neq 2, \ldots, X_6 \neq 2\}$

(b) T_2 is **NOT** a stopping time
   - $\{T_2 = n\} = \{X_{n+1} = 2, X_n \neq 2, \ldots, X_1 \neq 2\}$ dependes on X_{n+1}

(c) T_3 is a stopping time
   - $\{T_3 = n\} = \{X_{n-1} = 2, X_{n-2} \neq 2, \ldots, X_1 \neq 2\}$

(d) T_4 is a stopping time
   - $\{T_4 = n\} = \{T_1 = n - 1\} = \{X_{n-1} = 2, X_{n-2} \neq 2, \ldots, X_6 \neq 2\}$

(e) T_5 is a stopping time
   - $\{T_5 = n\} = \{X_n = X_{n-1}, X_{n-1} \neq X_{n-2}, \ldots, X_{10} \neq X_9\}$

(f) T_6 is **NOT** a stopping time
   - $\{T_6 = 1\} = \{X_1 = X_5\}$ dependes on X_5

(g) T_7 is a stopping time
   - Since $\{T_7 = n\}$ does not depends on $X_i, \forall i \in \mathbb{N}$

Question 5

3.4 3(d) **2 / 2**

✓ **- 0 pts** Correct

- No, because $\{R_2 = n\} = \{X_n = 2, X_{n+1} \neq 2, X_{n+2} \neq 2, \ldots\}$ depends on $X_{n+1}, X_{n+2}, \ldots$

Part (d)

- Since $X_{R_2} = 2$ and $R_2 = 8$, the last visit to state 2 is X_8
- If we go to state 1 or 2, we must go back to state 2 again
- Therefore, the next state must be 4
- i.e. $\mathbb{P}_3\left(X_{R_2+1} = 4 \middle| X_{R_2} = 2, R_2 = 8\right) = 1$

Question 4

4. Consider a Markov chain with state space $S = \{1, 2\}$, with transition matrix

$$P = \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}.$$

Decide which of the following is a stopping time:

(a) $T_1 = \min\{n \geq 6 : X_n = 2\}$
(b) $T_2 = \min\{n \geq 1 : X_{n+1} = 2\}$
(c) $T_3 = \min\{n \geq 2 : X_{n-1} = 2\}$
(d) $T_4 = \min\{n > T_1 : X_{n-1} = 2\}$
(e) $T_5 = \min\{n \geq 10 : X_n = X_{n-1}\}$
(f) $T_6 = \min\{n \geq 1 : X_n = X_5\}$
(g) $T_7 = 10$

(a) T_1 is a stopping time
  - $\{T_1 = n\} = \{X_n = 2, X_{n-1} \neq 2, \ldots, X_6 \neq 2\}$

(b) T_2 is **NOT** a stopping time
  - $\{T_2 = n\} = \{X_{n+1} = 2, X_n \neq 2, \ldots, X_1 \neq 2\}$ dependes on X_{n+1}

(c) T_3 is a stopping time
  - $\{T_3 = n\} = \{X_{n-1} = 2, X_{n-2} \neq 2, \ldots, X_1 \neq 2\}$

(d) T_4 is a stopping time
  - $\{T_4 = n\} = \{T_1 = n - 1\} = \{X_{n-1} = 2, X_{n-2} \neq 2, \ldots, X_6 \neq 2\}$

(e) T_5 is a stopping time
  - $\{T_5 = n\} = \{X_n = X_{n-1}, X_{n-1} \neq X_{n-2}, \ldots, X_{10} \neq X_9\}$

(f) T_6 is **NOT** a stopping time
  - $\{T_6 = 1\} = \{X_1 = X_5\}$ dependes on X_5

(g) T_7 is a stopping time
  - Since $\{T_7 = n\}$ does not depends on $X_i, \forall i \in \mathbb{N}$

Question 5

4.1 4(a) **2 / 2**

✓ **- 0 pts** Correct

- No, because $\{R_2 = n\} = \{X_n = 2, X_{n+1} \neq 2, X_{n+2} \neq 2, \dots\}$ depends on $X_{n+1}, X_{n+2}, \dots$

Part (d)

- Since $X_{R_2} = 2$ and $R_2 = 8$, the last visit to state 2 is X_8
- If we go to state 1 or 2, we must go back to state 2 again
- Therefore, the next state must be 4
- i.e. $\mathbb{P}_3\left(X_{R_2+1} = 4 \middle| X_{R_2} = 2, R_2 = 8\right) = 1$

Question 4

4. Consider a Markov chain with state space $S = \{1, 2\}$, with transition matrix

$$P = \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}.$$

Decide which of the following is a stopping time:

(a) $T_1 = \min\{n \geq 6 : X_n = 2\}$
(b) $T_2 = \min\{n \geq 1 : X_{n+1} = 2\}$
(c) $T_3 = \min\{n \geq 2 : X_{n-1} = 2\}$
(d) $T_4 = \min\{n > T_1 : X_{n-1} = 2\}$
(e) $T_5 = \min\{n \geq 10 : X_n = X_{n-1}\}$
(f) $T_6 = \min\{n \geq 1 : X_n = X_5\}$
(g) $T_7 = 10$

(a) T_1 is a stopping time
  - $\{T_1 = n\} = \{X_n = 2, X_{n-1} \neq 2, \dots, X_6 \neq 2\}$

(b) T_2 is **NOT** a stopping time
  - $\{T_2 = n\} = \{X_{n+1} = 2, X_n \neq 2, \dots, X_1 \neq 2\}$ dependes on X_{n+1}

(c) T_3 is a stopping time
  - $\{T_3 = n\} = \{X_{n-1} = 2, X_{n-2} \neq 2, \dots, X_1 \neq 2\}$

(d) T_4 is a stopping time
  - $\{T_4 = n\} = \{T_1 = n - 1\} = \{X_{n-1} = 2, X_{n-2} \neq 2, \dots, X_6 \neq 2\}$

(e) T_5 is a stopping time
  - $\{T_5 = n\} = \{X_n = X_{n-1}, X_{n-1} \neq X_{n-2}, \dots, X_{10} \neq X_9\}$

(f) T_6 is **NOT** a stopping time
  - $\{T_6 = 1\} = \{X_1 = X_5\}$ dependes on X_5

(g) T_7 is a stopping time
  - Since $\{T_7 = n\}$ does not depends on $X_i, \forall i \in \mathbb{N}$

Question 5

4.2 4(b) **2 / 2**

✓ **- 0 pts** Correct

- No, because $\{R_2 = n\} = \{X_n = 2, X_{n+1} \neq 2, X_{n+2} \neq 2, \dots\}$ depends on $X_{n+1}, X_{n+2}, \dots$

Part (d)

- Since $X_{R_2} = 2$ and $R_2 = 8$, the last visit to state 2 is X_8
- If we go to state 1 or 2, we must go back to state 2 again
- Therefore, the next state must be 4
- i.e. $\mathbb{P}_3\left(X_{R_2+1} = 4 \middle| X_{R_2} = 2, R_2 = 8\right) = 1$

Question 4

4. Consider a Markov chain with state space $S = \{1, 2\}$, with transition matrix

$$P = \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}.$$

Decide which of the following is a stopping time:

(a) $T_1 = \min\{n \geq 6 : X_n = 2\}$
(b) $T_2 = \min\{n \geq 1 : X_{n+1} = 2\}$
(c) $T_3 = \min\{n \geq 2 : X_{n-1} = 2\}$
(d) $T_4 = \min\{n > T_1 : X_{n-1} = 2\}$
(e) $T_5 = \min\{n \geq 10 : X_n = X_{n-1}\}$
(f) $T_6 = \min\{n \geq 1 : X_n = X_5\}$
(g) $T_7 = 10$

(a) T_1 is a stopping time
   - $\{T_1 = n\} = \{X_n = 2, X_{n-1} \neq 2, \dots, X_6 \neq 2\}$

(b) T_2 is **NOT** a stopping time
   - $\{T_2 = n\} = \{X_{n+1} = 2, X_n \neq 2, \dots, X_1 \neq 2\}$ dependes on X_{n+1}

(c) T_3 is a stopping time
   - $\{T_3 = n\} = \{X_{n-1} = 2, X_{n-2} \neq 2, \dots, X_1 \neq 2\}$

(d) T_4 is a stopping time
   - $\{T_4 = n\} = \{T_1 = n - 1\} = \{X_{n-1} = 2, X_{n-2} \neq 2, \dots, X_6 \neq 2\}$

(e) T_5 is a stopping time
   - $\{T_5 = n\} = \{X_n = X_{n-1}, X_{n-1} \neq X_{n-2}, \dots, X_{10} \neq X_9\}$

(f) T_6 is **NOT** a stopping time
   - $\{T_6 = 1\} = \{X_1 = X_5\}$ dependes on X_5

(g) T_7 is a stopping time
   - Since $\{T_7 = n\}$ does not depends on $X_i, \forall i \in \mathbb{N}$

Question 5

4.3 4(c) **2 / 2**

✓ **- 0 pts** Correct

- No, because $\{R_2 = n\} = \{X_n = 2, X_{n+1} \neq 2, X_{n+2} \neq 2, \dots\}$ depends on $X_{n+1}, X_{n+2}, \dots$

Part (d)

- Since $X_{R_2} = 2$ and $R_2 = 8$, the last visit to state 2 is X_8
- If we go to state 1 or 2, we must go back to state 2 again
- Therefore, the next state must be 4
- i.e. $\mathbb{P}_3\left(X_{R_2+1} = 4 \middle| X_{R_2} = 2, R_2 = 8\right) = 1$

Question 4

4. Consider a Markov chain with state space $S = \{1, 2\}$, with transition matrix

$$P = \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}.$$

Decide which of the following is a stopping time:

(a) $T_1 = \min\{n \geq 6 : X_n = 2\}$
(b) $T_2 = \min\{n \geq 1 : X_{n+1} = 2\}$
(c) $T_3 = \min\{n \geq 2 : X_{n-1} = 2\}$
(d) $T_4 = \min\{n > T_1 : X_{n-1} = 2\}$
(e) $T_5 = \min\{n \geq 10 : X_n = X_{n-1}\}$
(f) $T_6 = \min\{n \geq 1 : X_n = X_5\}$
(g) $T_7 = 10$

(a) T_1 is a stopping time
- $\{T_1 = n\} = \{X_n = 2, X_{n-1} \neq 2, \dots, X_6 \neq 2\}$

(b) T_2 is **NOT** a stopping time
- $\{T_2 = n\} = \{X_{n+1} = 2, X_n \neq 2, \dots, X_1 \neq 2\}$ dependes on X_{n+1}

(c) T_3 is a stopping time
- $\{T_3 = n\} = \{X_{n-1} = 2, X_{n-2} \neq 2, \dots, X_1 \neq 2\}$

(d) T_4 is a stopping time
- $\{T_4 = n\} = \{T_1 = n - 1\} = \{X_{n-1} = 2, X_{n-2} \neq 2, \dots, X_6 \neq 2\}$

(e) T_5 is a stopping time
- $\{T_5 = n\} = \{X_n = X_{n-1}, X_{n-1} \neq X_{n-2}, \dots, X_{10} \neq X_9\}$

(f) T_6 is **NOT** a stopping time
- $\{T_6 = 1\} = \{X_1 = X_5\}$ dependes on X_5

(g) T_7 is a stopping time
- Since $\{T_7 = n\}$ does not depends on $X_i, \forall i \in \mathbb{N}$

Question 5

4.4 4(d) **2 / 2**

✓ **- 0 pts** Correct

- No, because $\{R_2 = n\} = \{X_n = 2, X_{n+1} \neq 2, X_{n+2} \neq 2, \dots\}$ depends on $X_{n+1}, X_{n+2}, \dots$

Part (d)

- Since $X_{R_2} = 2$ and $R_2 = 8$, the last visit to state 2 is X_8
- If we go to state 1 or 2, we must go back to state 2 again
- Therefore, the next state must be 4
- i.e. $\mathbb{P}_3\left(X_{R_2+1} = 4 \middle| X_{R_2} = 2, R_2 = 8\right) = 1$

Question 4

4. Consider a Markov chain with state space $S = \{1, 2\}$, with transition matrix

$$P = \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}.$$

Decide which of the following is a stopping time:

(a) $T_1 = \min\{n \geq 6 : X_n = 2\}$
(b) $T_2 = \min\{n \geq 1 : X_{n+1} = 2\}$
(c) $T_3 = \min\{n \geq 2 : X_{n-1} = 2\}$
(d) $T_4 = \min\{n > T_1 : X_{n-1} = 2\}$
(e) $T_5 = \min\{n \geq 10 : X_n = X_{n-1}\}$
(f) $T_6 = \min\{n \geq 1 : X_n = X_5\}$
(g) $T_7 = 10$

(a) T_1 is a stopping time
  - $\{T_1 = n\} = \{X_n = 2, X_{n-1} \neq 2, \dots, X_6 \neq 2\}$

(b) T_2 is **NOT** a stopping time
  - $\{T_2 = n\} = \{X_{n+1} = 2, X_n \neq 2, \dots, X_1 \neq 2\}$ dependes on X_{n+1}

(c) T_3 is a stopping time
  - $\{T_3 = n\} = \{X_{n-1} = 2, X_{n-2} \neq 2, \dots, X_1 \neq 2\}$

(d) T_4 is a stopping time
  - $\{T_4 = n\} = \{T_1 = n - 1\} = \{X_{n-1} = 2, X_{n-2} \neq 2, \dots, X_6 \neq 2\}$

(e) T_5 is a stopping time
  - $\{T_5 = n\} = \{X_n = X_{n-1}, X_{n-1} \neq X_{n-2}, \dots, X_{10} \neq X_9\}$

(f) T_6 is **NOT** a stopping time
  - $\{T_6 = 1\} = \{X_1 = X_5\}$ dependes on X_5

(g) T_7 is a stopping time
  - Since $\{T_7 = n\}$ does not depends on $X_i, \forall i \in \mathbb{N}$

Question 5

4.5 4(e) **2 / 2**

✓ **- 0 pts** Correct

- No, because $\{R_2 = n\} = \{X_n = 2, X_{n+1} \neq 2, X_{n+2} \neq 2, \ldots\}$ depends on $X_{n+1}, X_{n+2}, \ldots$

Part (d)

- Since $X_{R_2} = 2$ and $R_2 = 8$, the last visit to state 2 is X_8
- If we go to state 1 or 2, we must go back to state 2 again
- Therefore, the next state must be 4
- i.e. $\mathbb{P}_3\left(X_{R_2+1} = 4 \middle| X_{R_2} = 2, R_2 = 8\right) = 1$

Question 4

4. Consider a Markov chain with state space $S = \{1, 2\}$, with transition matrix

$$P = \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}.$$

Decide which of the following is a stopping time:

(a) $T_1 = \min\{n \geq 6 : X_n = 2\}$
(b) $T_2 = \min\{n \geq 1 : X_{n+1} = 2\}$
(c) $T_3 = \min\{n \geq 2 : X_{n-1} = 2\}$
(d) $T_4 = \min\{n > T_1 : X_{n-1} = 2\}$
(e) $T_5 = \min\{n \geq 10 : X_n = X_{n-1}\}$
(f) $T_6 = \min\{n \geq 1 : X_n = X_5\}$
(g) $T_7 = 10$

(a) T_1 is a stopping time
   - $\{T_1 = n\} = \{X_n = 2, X_{n-1} \neq 2, \ldots, X_6 \neq 2\}$

(b) T_2 is **NOT** a stopping time
   - $\{T_2 = n\} = \{X_{n+1} = 2, X_n \neq 2, \ldots, X_1 \neq 2\}$ dependes on X_{n+1}

(c) T_3 is a stopping time
   - $\{T_3 = n\} = \{X_{n-1} = 2, X_{n-2} \neq 2, \ldots, X_1 \neq 2\}$

(d) T_4 is a stopping time
   - $\{T_4 = n\} = \{T_1 = n - 1\} = \{X_{n-1} = 2, X_{n-2} \neq 2, \ldots, X_6 \neq 2\}$

(e) T_5 is a stopping time
   - $\{T_5 = n\} = \{X_n = X_{n-1}, X_{n-1} \neq X_{n-2}, \ldots, X_{10} \neq X_9\}$

(f) T_6 is **NOT** a stopping time
   - $\{T_6 = 1\} = \{X_1 = X_5\}$ dependes on X_5

(g) T_7 is a stopping time
   - Since $\{T_7 = n\}$ does not depends on $X_i, \forall i \in \mathbb{N}$

Question 5

4.6 4(f) **2 / 2**

✓ **- 0 pts** Correct

- No, because $\{R_2 = n\} = \{X_n = 2, X_{n+1} \neq 2, X_{n+2} \neq 2, \dots\}$ depends on $X_{n+1}, X_{n+2}, \dots$

Part (d)

- Since $X_{R_2} = 2$ and $R_2 = 8$, the last visit to state 2 is X_8
- If we go to state 1 or 2, we must go back to state 2 again
- Therefore, the next state must be 4
- i.e. $\mathbb{P}_3\left(X_{R_2+1} = 4 \middle| X_{R_2} = 2, R_2 = 8\right) = 1$

Question 4

4. Consider a Markov chain with state space $S = \{1, 2\}$, with transition matrix

$$P = \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}.$$

Decide which of the following is a stopping time:

(a) $T_1 = \min\{n \geq 6 : X_n = 2\}$
(b) $T_2 = \min\{n \geq 1 : X_{n+1} = 2\}$
(c) $T_3 = \min\{n \geq 2 : X_{n-1} = 2\}$
(d) $T_4 = \min\{n > T_1 : X_{n-1} = 2\}$
(e) $T_5 = \min\{n \geq 10 : X_n = X_{n-1}\}$
(f) $T_6 = \min\{n \geq 1 : X_n = X_5\}$
(g) $T_7 = 10$

(a) T_1 is a stopping time
   - $\{T_1 = n\} = \{X_n = 2, X_{n-1} \neq 2, \dots, X_6 \neq 2\}$

(b) T_2 is **NOT** a stopping time
   - $\{T_2 = n\} = \{X_{n+1} = 2, X_n \neq 2, \dots, X_1 \neq 2\}$ dependes on X_{n+1}

(c) T_3 is a stopping time
   - $\{T_3 = n\} = \{X_{n-1} = 2, X_{n-2} \neq 2, \dots, X_1 \neq 2\}$

(d) T_4 is a stopping time
   - $\{T_4 = n\} = \{T_1 = n - 1\} = \{X_{n-1} = 2, X_{n-2} \neq 2, \dots, X_6 \neq 2\}$

(e) T_5 is a stopping time
   - $\{T_5 = n\} = \{X_n = X_{n-1}, X_{n-1} \neq X_{n-2}, \dots, X_{10} \neq X_9\}$

(f) T_6 is **NOT** a stopping time
   - $\{T_6 = 1\} = \{X_1 = X_5\}$ dependes on X_5

(g) T_7 is a stopping time
   - Since $\{T_7 = n\}$ does not depends on $X_i, \forall i \in \mathbb{N}$

Question 5

4.7 4(g) 2 / 2

✓ **- 0 pts** Correct

1.8. Consider the following transition matrices. Identify the transient and recurrent states, and the irreducible closed sets in the Markov chains. Give reasons for your answers.

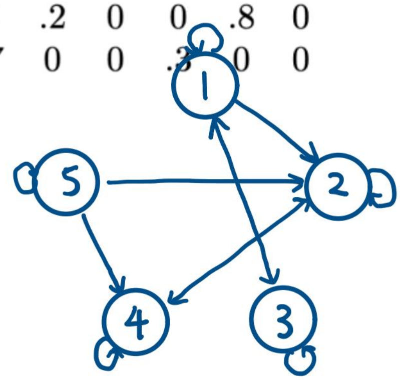
(a)	**1**	**2**	**3**	**4**	**5**
1	.4	.3	.3	0	0
2	0	.5	0	.5	0
3	.5	0	.5	0	0
4	0	.5	0	.5	0
5	0	.3	0	.3	.4

(b)	**1**	**2**	**3**	**4**	**5**	**6**
1	.1	0	0	.4	.5	0
2	.1	.2	.2	0	.5	0
3	0	.1	.3	0	0	.6
4	.1	0	0	.9	0	0
5	0	0	0	.4	0	.6
6	0	0	0	0	.5	.5

(c)	**1**	**2**	**3**	**4**	**5**
1	0	0	0	0	1
2	0	.2	0	.8	0
3	.1	.2	.3	.4	0
4	0	.6	0	.4	0
5	.3	0	0	0	.7

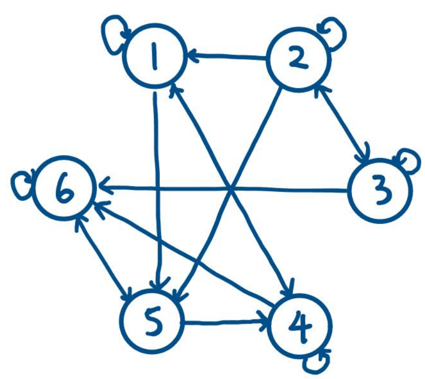
(d)	**1**	**2**	**3**	**4**	**5**	**6**
1	.8	0	0	.2	0	0
2	0	.5	0	0	.5	0
3	0	0	.3	.4	.3	0
4	.1	0	0	.9	0	0
5	0	.2	0	0	.8	0
6	.7	0	0	.3	0	0

Part (a)



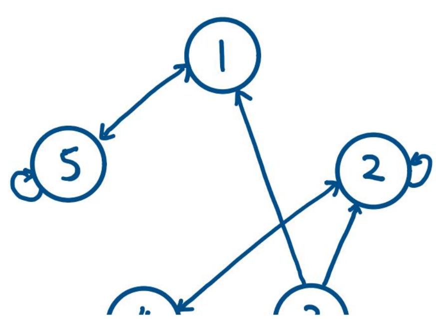
- $1 \Rightarrow 2 \not\Rightarrow 1$, so 1 is transient
- $3 \Rightarrow 2 \not\Rightarrow 3$, so 3 is transient
- $5 \Rightarrow 4 \not\Rightarrow 5$, so 5 is transient
- $\{2,4\}$ is a irreducible closed set, since
  - $\{2,4\} \Rightarrow \{2,4\}$
  - $\{2,4\} \not\Rightarrow \{1,3,5\}$
- Therefore 2, 4 are recurrent

Part (b)



- $2 \Rightarrow 5 \not\Rightarrow 2$, so 2 is transient
- $3 \Rightarrow 5 \not\Rightarrow 3$, so 3 is transient
- $\{1,4,5,6\}$ is a irreducible closed set, since
  - $\{1,4,5,6\} \Rightarrow \{1,4,5,6\}$
  - $\{1,4,5,6\} \not\Rightarrow \{2,3\}$
- Therefore 1, 4, 5, 6 are recurrent

Part (c)



- $3 \Rightarrow 1 \not\Rightarrow 3$, so 3 is transient
- $\{1,5\}$ is a irreducible closed set, since
  - $\{1,5\} \Rightarrow \{1,5\}$
  - $\{1,5\} \not\Rightarrow \{2,3,4\}$
- $\{2,4\}$ is a irreducible closed set, since

5.1 1.8(a) **2 / 2**

✓ **- 0 pts** Correct

1.8. Consider the following transition matrices. Identify the transient and recurrent states, and the irreducible closed sets in the Markov chains. Give reasons for your answers.

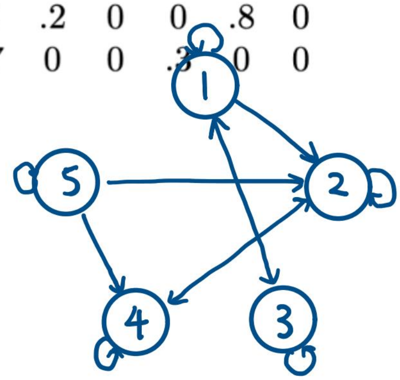
(a)	**1**	**2**	**3**	**4**	**5**
1	.4	.3	.3	0	0
2	0	.5	0	.5	0
3	.5	0	.5	0	0
4	0	.5	0	.5	0
5	0	.3	0	.3	.4

(b)	**1**	**2**	**3**	**4**	**5**	**6**
1	.1	0	0	.4	.5	0
2	.1	.2	.2	0	.5	0
3	0	.1	.3	0	0	.6
4	.1	0	0	.9	0	0
5	0	0	0	.4	0	.6
6	0	0	0	0	.5	.5

(c)	**1**	**2**	**3**	**4**	**5**
1	0	0	0	0	1
2	0	.2	0	.8	0
3	.1	.2	.3	.4	0
4	0	.6	0	.4	0
5	.3	0	0	0	.7

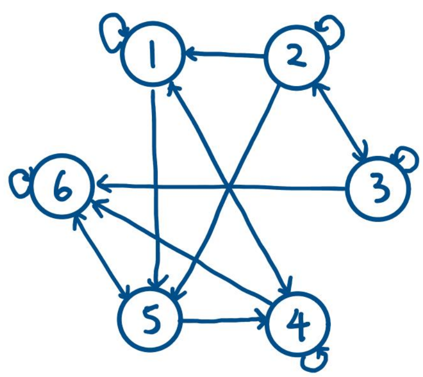
(d)	**1**	**2**	**3**	**4**	**5**	**6**
1	.8	0	0	.2	0	0
2	0	.5	0	0	.5	0
3	0	0	.3	.4	.3	0
4	.1	0	0	.9	0	0
5	0	.2	0	0	.8	0
6	.7	0	0	.3	0	0

Part (a)



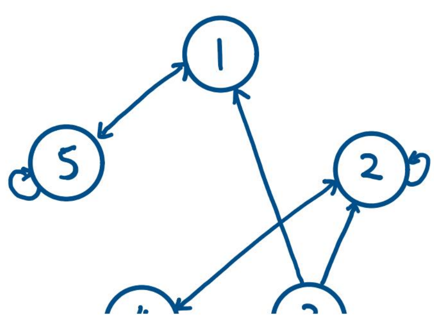
- $1 \Rightarrow 2 \not\Rightarrow 1$, so 1 is transient
- $3 \Rightarrow 2 \not\Rightarrow 3$, so 3 is transient
- $5 \Rightarrow 4 \not\Rightarrow 5$, so 5 is transient
- $\{2,4\}$ is a irreducible closed set, since
  - $\{2,4\} \Rightarrow \{2,4\}$
  - $\{2,4\} \not\Rightarrow \{1,3,5\}$
- Therefore 2, 4 are recurrent

Part (b)



- $2 \Rightarrow 5 \not\Rightarrow 2$, so 2 is transient
- $3 \Rightarrow 5 \not\Rightarrow 3$, so 3 is transient
- $\{1,4,5,6\}$ is a irreducible closed set, since
  - $\{1,4,5,6\} \Rightarrow \{1,4,5,6\}$
  - $\{1,4,5,6\} \not\Rightarrow \{2,3\}$
- Therefore 1, 4, 5, 6 are recurrent

Part (c)



- $3 \Rightarrow 1 \not\Rightarrow 3$, so 3 is transient
- $\{1,5\}$ is a irreducible closed set, since
  - $\{1,5\} \Rightarrow \{1,5\}$
  - $\{1,5\} \not\Rightarrow \{2,3,4\}$
- $\{2,4\}$ is a irreducible closed set, since

5.2 1.8(b) **2 / 2**

✓ **- 0 pts** Correct

1.8. Consider the following transition matrices. Identify the transient and recurrent states, and the irreducible closed sets in the Markov chains. Give reasons for your answers.

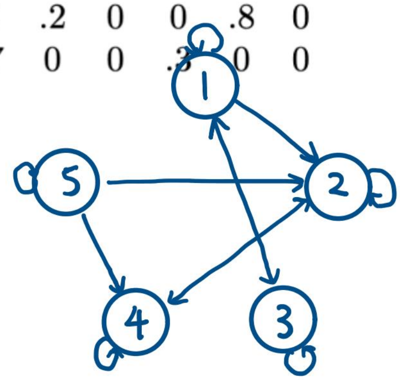
(a)	**1**	**2**	**3**	**4**	**5**
1	.4	.3	.3	0	0
2	0	.5	0	.5	0
3	.5	0	.5	0	0
4	0	.5	0	.5	0
5	0	.3	0	.3	.4

(b)	**1**	**2**	**3**	**4**	**5**	**6**
1	.1	0	0	.4	.5	0
2	.1	.2	.2	0	.5	0
3	0	.1	.3	0	0	.6
4	.1	0	0	.9	0	0
5	0	0	0	.4	0	.6
6	0	0	0	0	.5	.5

(c)	**1**	**2**	**3**	**4**	**5**
1	0	0	0	0	1
2	0	.2	0	.8	0
3	.1	.2	.3	.4	0
4	0	.6	0	.4	0
5	.3	0	0	0	.7

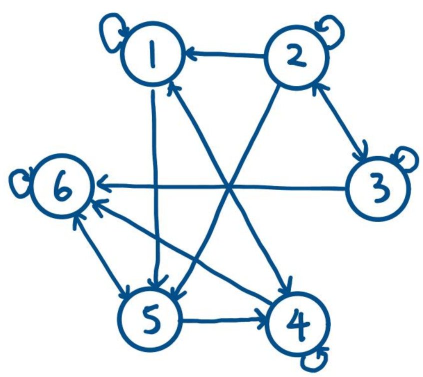
(d)	**1**	**2**	**3**	**4**	**5**	**6**
1	.8	0	0	.2	0	0
2	0	.5	0	0	.5	0
3	0	0	.3	.4	.3	0
4	.1	0	0	.9	0	0
5	0	.2	0	0	.8	0
6	.7	0	0	.3	0	0

Part (a)



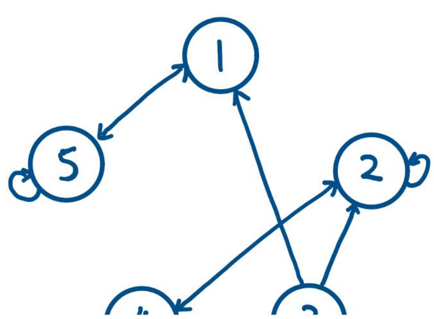
- $1 \Rightarrow 2 \nRightarrow 1$, so 1 is transient
- $3 \Rightarrow 2 \nRightarrow 3$, so 3 is transient
- $5 \Rightarrow 4 \nRightarrow 5$, so 5 is transient
- $\{2,4\}$ is a irreducible closed set, since
  - $\{2,4\} \Rightarrow \{2,4\}$
  - $\{2,4\} \nRightarrow \{1,3,5\}$
- Therefore 2, 4 are recurrent

Part (b)



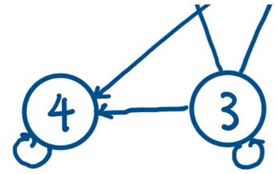
- $2 \Rightarrow 5 \nRightarrow 2$, so 2 is transient
- $3 \Rightarrow 5 \nRightarrow 3$, so 3 is transient
- $\{1,4,5,6\}$ is a irreducible closed set, since
  - $\{1,4,5,6\} \Rightarrow \{1,4,5,6\}$
  - $\{1,4,5,6\} \nRightarrow \{2,3\}$
- Therefore 1, 4, 5, 6 are recurrent

Part (c)



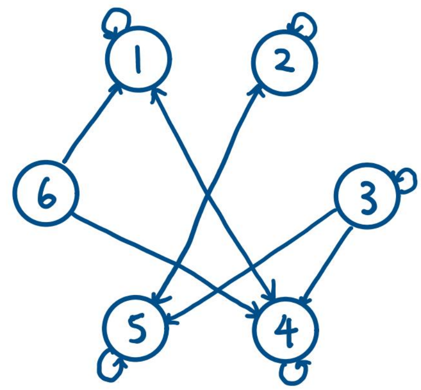
- $3 \Rightarrow 1 \nRightarrow 3$, so 3 is transient
- $\{1,5\}$ is a irreducible closed set, since
  - $\{1,5\} \Rightarrow \{1,5\}$
  - $\{1,5\} \nRightarrow \{2,3,4\}$
- $\{2,4\}$ is a irreducible closed set, since

- $\{1,5\} \not\Rightarrow \{2,3,4\}$



- $\{2,4\}$ is a irreducible closed set, since
  - $\{2,4\} \Rightarrow \{2,4\}$
  - $\{2,4\} \not\Rightarrow \{1,3,5\}$
- Therefore 1, 2, 4, 5 are recurrent

Part (d)



- $3 \Rightarrow 5 \not\Rightarrow 3$, so 3 is transient
- $6 \Rightarrow 1 \not\Rightarrow 6$, so 6 is transient
- $\{1,4\}$ is a irreducible closed set, since
  - $\{1,4\} \Rightarrow \{1,4\}$
  - $\{1,4\} \not\Rightarrow \{2,3,5,6\}$
- $\{2,5\}$ is a irreducible closed set, since
  - $\{2,5\} \Rightarrow \{2,5\}$
  - $\{2,5\} \not\Rightarrow \{1,3,4,6\}$
- Therefore 1, 2, 4, 5 are recurrent

6. Consider a discrete time Markov chain with state space $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ and one step transition probabilities given by

$$\begin{aligned} p(i, i+1) &= p \qquad \text{for all } i \in \mathbb{Z}; \\ p(i, i-1) &= q \qquad \text{for all } i \in \mathbb{Z} \end{aligned}$$

and zero otherwise, with $p, q \geq 0$ and $p + q = 1$.

(a) Assume that $p = 0$. Find all the closed sets.

(b) Assume that $p = 0$. Find all the irreducible sets.

(c) Now assume that $p, q > 0$. Prove that either all the states are recurrent, or all the states are transient.

Part (a)

- $A_n = \{\dots, n-2, n-1, n\}, \forall n \in \mathbb{Z}$ are all the closed sets
- Since if $i \in A_n$, and $p(i, i-1) = q = 1$, then $i - 1 \in A_n$

Part (b)

- There is not irreducible set
- Since $p(i, i+1) = 0$, and $p(i, i-1) = 1$, $i \Rightarrow j$ and $i \Rightarrow j$ cannot be both true for all $i, j \in A_n$

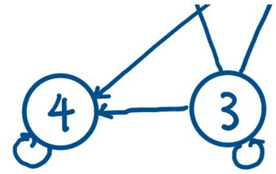
Part (c)

- Suppose, for sake of contradiction, that r is recurrent, and t is transient
- Let $n = |r - t|$, then $p^n(r,t) \geq \begin{cases} p^n & (\text{if } r < t) \\ q^n & (\text{if } r > t) \end{cases} > 0$, i.e. $r \Rightarrow t$
- By lamma 1.9, r is recurrent and $r \Rightarrow t$, so t is also transient
- Contradiction! Therefore either all the states are recurrent or transient

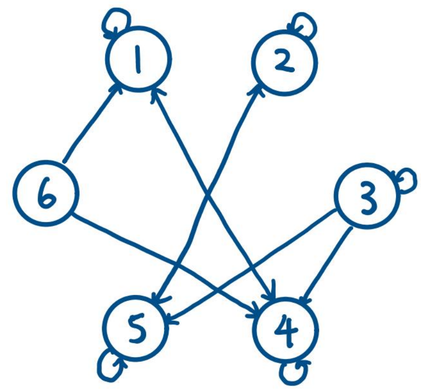
5.3 1.8(c) **2 / 2**

✓ **- 0 pts** Correct

- $\{1,5\} \nRightarrow \{2,3,4\}$
- $\{2,4\}$ is a irreducible closed set, since
    - $\{2,4\} \Rightarrow \{2,4\}$
    - $\{2,4\} \nRightarrow \{1,3,5\}$
- Therefore $1, 2, 4, 5$ are recurrent



Part (d)



- $3 \Rightarrow 5 \nRightarrow 3$, so 3 is transient
- $6 \Rightarrow 1 \nRightarrow 6$, so 6 is transient
- $\{1,4\}$ is a irreducible closed set, since
    - $\{1,4\} \Rightarrow \{1,4\}$
    - $\{1,4\} \nRightarrow \{2,3,5,6\}$
- $\{2,5\}$ is a irreducible closed set, since
    - $\{2,5\} \Rightarrow \{2,5\}$
    - $\{2,5\} \nRightarrow \{1,3,4,6\}$
- Therefore $1, 2, 4, 5$ are recurrent

6. Consider a discrete time Markov chain with state space $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ and one step transition probabilities given by

$$\begin{aligned} p(i, i+1) &= p \qquad \text{for all } i \in \mathbb{Z}; \\ p(i, i-1) &= q \qquad \text{for all } i \in \mathbb{Z} \end{aligned}$$

and zero otherwise, with $p, q \geq 0$ and $p + q = 1$.

(a) Assume that $p = 0$. Find all the closed sets.

(b) Assume that $p = 0$. Find all the irreducible sets.

(c) Now assume that $p, q > 0$. Prove that either all the states are recurrent, or all the states are transient.

Part (a)

- $A_n = \{\dots, n-2, n-1, n\}, \forall n \in \mathbb{Z}$ are all the closed sets
- Since if $i \in A_n$, and $p(i, i-1) = q = 1$, then $i - 1 \in A_n$

Part (b)

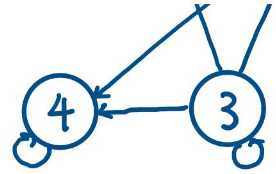
- There is not irreducible set
- Since $p(i, i+1) = 0$, and $p(i, i-1) = 1$, $i \Rightarrow j$ and $i \Rightarrow j$ cannot be both true for all $i, j \in A_n$

Part (c)

- Suppose, for sake of contradiction, that r is recurrent, and t is transient
- Let $n = |r - t|$, then $p^n(r,t) \geq \begin{cases} p^n & (\text{if } r < t) \\ q^n & (\text{if } r > t) \end{cases} > 0$, i.e. $r \Rightarrow t$
- By lamma 1.9, r is recurrent and $r \Rightarrow t$, so t is also transient
- Contradiction! Therefore either all the states are recurrent or transient

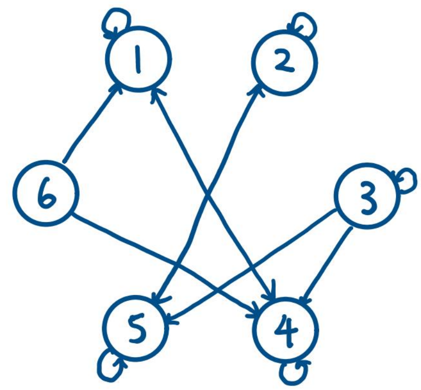
5.4 1.8(d) **2 / 2**

✓ **- 0 pts** Correct



- $\{1,5\} \nRightarrow \{2,3,4\}$
- $\{2,4\}$ is a irreducible closed set, since
  - $\{2,4\} \Rightarrow \{2,4\}$
  - $\{2,4\} \nRightarrow \{1,3,5\}$
- Therefore 1, 2, 4, 5 are recurrent

Part (d)



- $3 \Rightarrow 5 \nRightarrow 3$, so 3 is transient
- $6 \Rightarrow 1 \nRightarrow 6$, so 6 is transient
- $\{1,4\}$ is a irreducible closed set, since
  - $\{1,4\} \Rightarrow \{1,4\}$
  - $\{1,4\} \nRightarrow \{2,3,5,6\}$
- $\{2,5\}$ is a irreducible closed set, since
  - $\{2,5\} \Rightarrow \{2,5\}$
  - $\{2,5\} \nRightarrow \{1,3,4,6\}$
- Therefore 1, 2, 4, 5 are recurrent

6. Consider a discrete time Markov chain with state space $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and one step transition probabilities given by

$$\begin{aligned} p(i, i+1) &= p \qquad \text{for all } i \in \mathbb{Z}; \\ p(i, i-1) &= q \qquad \text{for all } i \in \mathbb{Z} \end{aligned}$$

and zero otherwise, with $p, q \geq 0$ and $p + q = 1$.

(a) Assume that $p = 0$. Find all the closed sets.

(b) Assume that $p = 0$. Find all the irreducible sets.

(c) Now assume that $p, q > 0$. Prove that either all the states are recurrent, or all the states are transient.

Part (a)

- $A_n = \{\ldots, n-2, n-1, n\}, \forall n \in \mathbb{Z}$ are all the closed sets
- Since if $i \in A_n$, and $p(i, i-1) = q = 1$, then $i - 1 \in A_n$

Part (b)

- There is not irreducible set
- Since $p(i, i+1) = 0$, and $p(i, i-1) = 1$, $i \Rightarrow j$ and $i \Rightarrow j$ cannot be both true for all $i, j \in A_n$

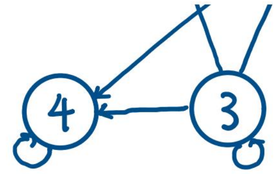
Part (c)

- Suppose, for sake of contradiction, that r is recurrent, and t is transient
- Let $n = |r - t|$, then $p^n(r,t) \geq \begin{cases} p^n & (\text{if } r < t) \\ q^n & (\text{if } r > t) \end{cases} > 0$, i.e. $r \Rightarrow t$
- By lamma 1.9, r is recurrent and $r \Rightarrow t$, so t is also transient
- Contradiction! Therefore either all the states are recurrent or transient

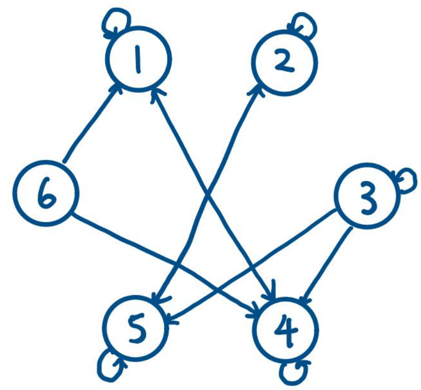
6.1 6(a) **4 / 4**

✓ **- 0 pts** Correct

- $\{1,5\} \nRightarrow \{2,3,4\}$
- $\{2,4\}$ is a irreducible closed set, since
  - $\{2,4\} \Rightarrow \{2,4\}$
  - $\{2,4\} \nRightarrow \{1,3,5\}$
- Therefore 1, 2, 4, 5 are recurrent



Part (d)



- $3 \Rightarrow 5 \nRightarrow 3$, so 3 is transient
- $6 \Rightarrow 1 \nRightarrow 6$, so 6 is transient
- $\{1,4\}$ is a irreducible closed set, since
  - $\{1,4\} \Rightarrow \{1,4\}$
  - $\{1,4\} \nRightarrow \{2,3,5,6\}$
- $\{2,5\}$ is a irreducible closed set, since
  - $\{2,5\} \Rightarrow \{2,5\}$
  - $\{2,5\} \nRightarrow \{1,3,4,6\}$
- Therefore 1, 2, 4, 5 are recurrent

6. Consider a discrete time Markov chain with state space $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and one step transition probabilities given by

$$p(i, i+1) = p \qquad \text{for all } i \in \mathbb{Z};$$
$$p(i, i-1) = q \qquad \text{for all } i \in \mathbb{Z}$$

and zero otherwise, with $p, q \geq 0$ and $p + q = 1$.

(a) Assume that $p = 0$. Find all the closed sets.

(b) Assume that $p = 0$. Find all the irreducible sets.

(c) Now assume that $p, q > 0$. Prove that either all the states are recurrent, or all the states are transient.

Part (a)

- $A_n = \{\ldots, n-2, n-1, n\}, \forall n \in \mathbb{Z}$ are all the closed sets
- Since if $i \in A_n$, and $p(i, i-1) = q = 1$, then $i - 1 \in A_n$

Part (b)

- There is not irreducible set
- Since $p(i, i+1) = 0$, and $p(i, i-1) = 1$, $i \Rightarrow j$ and $i \Rightarrow j$ cannot be both true for all $i, j \in A_n$

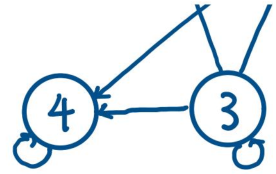
Part (c)

- Suppose, for sake of contradiction, that r is recurrent, and t is transient
- Let $n = |r - t|$, then $p^n(r,t) \geq \begin{cases} p^n & (\text{if } r < t) \\ q^n & (\text{if } r > t) \end{cases} > 0$, i.e. $r \Rightarrow t$
- By lamma 1.9, r is recurrent and $r \Rightarrow t$, so t is also transient
- Contradiction! Therefore either all the states are recurrent or transient

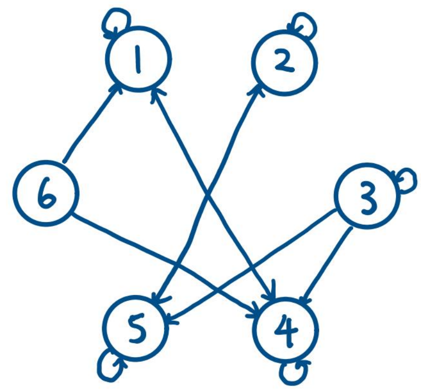
6.2 6(b) 2 / 3

✓ **- 1 pts** **The only irreducible sets are singletons**

- ○ $\{1,5\} \nRightarrow \{2,3,4\}$
- $\{2,4\}$ is a irreducible closed set, since
  - ○ $\{2,4\} \Rightarrow \{2,4\}$
  - ○ $\{2,4\} \nRightarrow \{1,3,5\}$
- Therefore $1, 2, 4, 5$ are recurrent



Part (d)



- $3 \Rightarrow 5 \nRightarrow 3$, so 3 is transient
- $6 \Rightarrow 1 \nRightarrow 6$, so 6 is transient
- $\{1,4\}$ is a irreducible closed set, since
  - ○ $\{1,4\} \Rightarrow \{1,4\}$
  - ○ $\{1,4\} \nRightarrow \{2,3,5,6\}$
- $\{2,5\}$ is a irreducible closed set, since
  - ○ $\{2,5\} \Rightarrow \{2,5\}$
  - ○ $\{2,5\} \nRightarrow \{1,3,4,6\}$
- Therefore $1, 2, 4, 5$ are recurrent

6. Consider a discrete time Markov chain with state space $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and one step transition probabilities given by

$$p(i, i+1) = p \qquad \text{for all } i \in \mathbb{Z};$$
$$p(i, i-1) = q \qquad \text{for all } i \in \mathbb{Z}$$

and zero otherwise, with $p, q \geq 0$ and $p + q = 1$.

(a) Assume that $p = 0$. Find all the closed sets.

(b) Assume that $p = 0$. Find all the irreducible sets.

(c) Now assume that $p, q > 0$. Prove that either all the states are recurrent, or all the states are transient.

Part (a)

- $A_n = \{\ldots, n-2, n-1, n\}, \forall n \in \mathbb{Z}$ are all the closed sets
- Since if $i \in A_n$, and $p(i, i-1) = q = 1$, then $i - 1 \in A_n$

Part (b)

- There is not irreducible set
- Since $p(i, i+1) = 0$, and $p(i, i-1) = 1$, $i \Rightarrow j$ and $i \Rightarrow j$ cannot be both true for all $i, j \in A_n$

Part (c)

- Suppose, for sake of contradiction, that r is recurrent, and t is transient
- Let $n = |r - t|$, then $p^n(r,t) \geq \begin{cases} p^n & (\text{if } r < t) \\ q^n & (\text{if } r > t) \end{cases} > 0$, i.e. $r \Rightarrow t$
- By lamma 1.9, r is recurrent and $r \Rightarrow t$, so t is also transient
- Contradiction! Therefore either all the states are recurrent or transient

6.3 6(c) **3 / 3**

✓ **- 0 pts** Correct

7. Let $\{X_n\}_{n\geq 0}$ be a Markov chain with transition probability $\{p(x,y)\}_{x,y\in\mathcal{S}}$ with some countable state space $\mathcal{S}$. That is, the process satisfies

$$P(X_{n+1}=y\,|\,X_n=x_n,\ldots,X_0=x_0)=p(x_n,y) \tag{1}$$

for all states $x_0,\ldots,x_n,y$ such that the conditioning event has positive probability.

(a) Using (1) (and general properties of probability and conditional probability), show that for any $0<k\leq n$,

$$P(X_{n+1}=y\,|\,X_n=x_n,\ldots,X_k=x_k)=p(x_n,y) \tag{2}$$

whenever the conditioning event has positive probability.

(b) Using (2), show that

$$P(X_{n-1}=x,X_{n+1}=z\,|\,X_n=y)=P(X_{n-1}=x\,|\,X_n=y)\cdot P(X_{n+1}=z\,|\,X_n=y)$$

for all states x,y,z such that $P(X_n=y)>0$. This is a special case of the statement that for a Markov chain, *given the present, the past and the future are independent.*

Part (a)

- $\mathbb{P}(X_{n+1}=y|X_n=x_n,\ldots,X_0=x_0)$

$$=\mathbb{P}(X_{n+1}=y|X_n=x_n,\ldots,X_k=x_k,\ldots X_0=x_0)$$

$$=\frac{\mathbb{P}(X_{n+1}=y,X_n=x_n,\ldots,X_k=x_k,\ldots X_0=x_0)}{\mathbb{P}(X_n=x_n,\ldots,X_k=x_k,\ldots X_0=x_0)}$$

$$=\frac{\mathbb{P}(X_{n+1}=y,X_n=x_n,\ldots,X_k=x_k)\prod_{l=0}^{k-1}\mathbb{P}(X_{l+1}=x_{l+1}|X_0=x_0,X_1=x_1\ldots,X_l=x_l)}{\mathbb{P}(X_n=x_n,\ldots,X_k=x_k)\prod_{l=0}^{k-1}\mathbb{P}(X_{l+1}=x_{l+1}|X_0=x_0,X_1=x_1\ldots,X_l=x_l)}$$

$$=\frac{\mathbb{P}(X_{n+1}=y,X_n=x_n,\ldots,X_k=x_k)}{\mathbb{P}(X_n=x_n,\ldots,X_k=x_k)}$$

$$=\mathbb{P}(X_{n+1}=y|X_n=x_n,\ldots,X_k=x_k)$$

- Therefore, $\mathbb{P}(X_{n+1}=y|X_n=x_n,\ldots,X_k=x_k)=p(x_n,y)$

Part (b)

- $\mathbb{P}(X_{n-1}=x,X_{n+1}=z|X_n=y)$

$$=\frac{\mathbb{P}(X_{n-1}=x,X_n=y,X_{n+1}=z)}{\mathbb{P}(X_n=y)}$$

$$=\frac{\mathbb{P}(X_{n-1}=x)\mathbb{P}(X_n=y|X_{n-1}=x)\mathbb{P}(X_{n+1}=z|X_n=y,X_{n-1}=x)}{\mathbb{P}(X_n=y)}$$

$$=\frac{\mathbb{P}(X_{n-1}=x)}{\mathbb{P}(X_n=y)}p(x,y)p(y,z)$$

- $\mathbb{P}(X_{n-1}=x|X_n=y)\mathbb{P}(X_{n+1}=z|X_n=y)$

$$=\frac{\mathbb{P}(X_{n-1}=x,X_n=y)\mathbb{P}(X_n=y,X_{n+1}=z)}{\left(\mathbb{P}(X_n=y)\right)^2}$$

7.1 7(a) 3 / 5

✓ - 2 pts The definition of conditional probability is not correctly used

7. Let $\{X_n\}_{n\geq 0}$ be a Markov chain with transition probability $\{p(x,y)\}_{x,y\in\mathcal{S}}$ with some countable state space $\mathcal{S}$. That is, the process satisfies

$$P(X_{n+1}=y\,|\,X_n=x_n,\ldots,X_0=x_0)=p(x_n,y) \tag{1}$$

for all states $x_0,\ldots,x_n,y$ such that the conditioning event has positive probability.

(a) Using (1) (and general properties of probability and conditional probability), show that for any $0<k\leq n$,

$$P(X_{n+1}=y\,|\,X_n=x_n,\ldots,X_k=x_k)=p(x_n,y) \tag{2}$$

whenever the conditioning event has positive probability.

(b) Using (2), show that

$$P(X_{n-1}=x,X_{n+1}=z\,|\,X_n=y)=P(X_{n-1}=x\,|\,X_n=y)\cdot P(X_{n+1}=z\,|\,X_n=y)$$

for all states x,y,z such that $P(X_n=y)>0$. This is a special case of the statement that for a Markov chain, *given the present, the past and the future are independent.*

Part (a)

- $\mathbb{P}(X_{n+1}=y|X_n=x_n,\ldots,X_0=x_0)$

$$=\mathbb{P}(X_{n+1}=y|X_n=x_n,\ldots,X_k=x_k,\ldots X_0=x_0)$$

$$=\frac{\mathbb{P}(X_{n+1}=y,X_n=x_n,\ldots,X_k=x_k,\ldots X_0=x_0)}{\mathbb{P}(X_n=x_n,\ldots,X_k=x_k,\ldots X_0=x_0)}$$

$$=\frac{\mathbb{P}(X_{n+1}=y,X_n=x_n,\ldots,X_k=x_k)\prod_{l=0}^{k-1}\mathbb{P}(X_{l+1}=x_{l+1}|X_0=x_0,X_1=x_1\ldots,X_l=x_l)}{\mathbb{P}(X_n=x_n,\ldots,X_k=x_k)\prod_{l=0}^{k-1}\mathbb{P}(X_{l+1}=x_{l+1}|X_0=x_0,X_1=x_1\ldots,X_l=x_l)}$$

$$=\frac{\mathbb{P}(X_{n+1}=y,X_n=x_n,\ldots,X_k=x_k)}{\mathbb{P}(X_n=x_n,\ldots,X_k=x_k)}$$

$$=\mathbb{P}(X_{n+1}=y|X_n=x_n,\ldots,X_k=x_k)$$

- Therefore, $\mathbb{P}(X_{n+1}=y|X_n=x_n,\ldots,X_k=x_k)=p(x_n,y)$

Part (b)

- $\mathbb{P}(X_{n-1}=x,X_{n+1}=z|X_n=y)$

$$=\frac{\mathbb{P}(X_{n-1}=x,X_n=y,X_{n+1}=z)}{\mathbb{P}(X_n=y)}$$

$$=\frac{\mathbb{P}(X_{n-1}=x)\mathbb{P}(X_n=y|X_{n-1}=x)\mathbb{P}(X_{n+1}=z|X_n=y,X_{n-1}=x)}{\mathbb{P}(X_n=y)}$$

$$=\frac{\mathbb{P}(X_{n-1}=x)}{\mathbb{P}(X_n=y)}p(x,y)p(y,z)$$

- $\mathbb{P}(X_{n-1}=x|X_n=y)\mathbb{P}(X_{n+1}=z|X_n=y)$

$$=\frac{\mathbb{P}(X_{n-1}=x,X_n=y)\mathbb{P}(X_n=y,X_{n+1}=z)}{\left(\mathbb{P}(X_n=y)\right)^2}$$

$$= \frac{\left(\mathbb{P}(X_n = y|X_{n-1} = x)\mathbb{P}(X_{n-1} = x)\right)\left(\mathbb{P}(X_{n+1} = z|X_n = y)\mathbb{P}(X_n = y)\right)}{\left(\mathbb{P}(X_n = y)\right)^2}$$

$$= \frac{p(x,y)\mathbb{P}(X_{n-1} = x)p(y,z)\mathbb{P}(X_n = y)}{\left(\mathbb{P}(X_n = y)\right)^2}$$

$$= \frac{\mathbb{P}(X_{n-1} = x)}{\mathbb{P}(X_n = y)}p(x,y)p(y,z)$$

- Therefore, $\mathbb{P}(X_{n-1} = x, X_{n+1} = z|X_n = y) = \mathbb{P}(X_{n-1} = x|X_n = y)\mathbb{P}(X_{n+1} = z|X_n = y)$

Comments:

- Instead of hand computations, you can use Matlab (or any other software) to perform the matrix operations required to solve the exercises. UW-Madison students can download Matlab for free from software.wisc.edu. Alternatively, you can use Matlab on the computers located on campus.

  Instead of downloading Matlab, you can use the online tool Octave that takes Matlab type commands.

  The Matlab commands you need: To enter a matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, type

  ```
  A=[1 2 3; 4 5 6; 7 8 9]
  ```

  You can calculate the multiplication of two matrices A and B by `C = A * B` and you can calculate matrix powers by `A^n`.

7.2 7(b) **5 / 5**

✓ **- 0 pts** Correct