Math 632 004 HW05

Shawn Zhong

TOTAL POINTS

50 / 50

QUESTION 1

1 Durrett 2.4 10 / 10

✓ **- 0 pts** Correct

- 3 pts Minor errors. Examples: Incorrectly evaluated integrals, arithmetic or algebra mistakes

- 5 pts Substantial errors. Examples: integrals set up incorrectly

- 8 pts Serious errors. Examples: does not set up any appropriate integral, does not use exponential density at all

- 10 pts No work submitted

QUESTION 2

2 Durrett 2.7abc 10 / 10

✓ **- 0 pts** Correct

- 1 pts (a) Minor error. Examples: arithmetic or algebra

- 3 pts (a) Little progress to solution. Examples: did not use properties of exponential distribution in any way

- 1 pts (b) Minor errors. Examples: computational mistakes

- 2 pts (b) Substantial errors. Examples: integral not set up correctly

- 4 pts (b) Little progress to solution. Examples: did not use properties of exponential distribution in any way

- 1 pts (c) Minor errors. Examples: computational mistakes

- 3 pts (c) Little progress to solution. Examples: did not use properties of exponential distribution in any way

- 10 pts No work submitted

QUESTION 3

3 Durrett 2.9b 10 / 10

✓ **- 0 pts** Correct

- 3 pts Minor errors. Examples: arithmetic or algebra errors

- 5 pts Substantial errors. Examples: incorrect expression for Bob's total waiting time

- 8 pts Little progress is made towards the solution

- 10 pts No work submitted

QUESTION 4

4 Durrett 2.13 10 / 10

✓ **- 0 pts** Correct

- 1 pts (a) Minor computational error

- 2 pts (a) Substantial error. Examples: incorrectly set up probability in terms of the Poisson processes

- 4 pts (a) Little progress to the solution

- 1 pts (b) Minor computational error

- 3 pts (b) Substantial error

- 1 pts (c) Incorrect conclusion due to minor error

- 3 pts (c) Serious errors: Incorrect definition of independence

- 10 pts No work submitted

QUESTION 5

5 Problem 5 10 / 10

✓ **- 0 pts** Correct

- 3 pts Computational errors

- 5 pts Substantial error. Examples: Mistake in integral set up

- 8 pts Little progress to the solution

- 10 pts No work submitted

- 3 pts Forgot the summation

HW5 - Problem & Solution

Thursday, October 25, 2018 9:47 PM

Math 632 Lecture 004 (Shinault), Fall 2018, Homework 5

Due Thursday November 1 by 10:05am

Question 1

2.4. Copy machine 1 is in use now. Machine 2 will be turned on at time t. Suppose that the machines fail at rate λ_i. What is the probability that machine 2 is the first to fail?

- Let $T_1 \sim \text{Exp}(\lambda_1)$ and $T_2 \sim \text{Exp}(\lambda_2)$ be the runing time for two machines until they fail
- $\mathbb{P}(T_2 + t < T_1) = \int_0^\infty f_{T_2}(s)\mathbb{P}(T_1 > s + t)ds = \int_0^\infty \lambda_2 e^{-\lambda_2 s} \cdot e^{-\lambda_1(s+t)}\, ds = \frac{\lambda_2}{\lambda_1 + \lambda_2} e^{-\lambda_1 t}$

Question 2

Durrett Exercise 2.7 (a), (b), (c), page 93. (Ignore part (d).) In part (b), compute first $P(V - U > s)$ for $s > 0$ either by integrating densities of S and T or by conditioning on the events $S < T$ and $T < S$. From $P(V - U > s)$ deduce the density function f_{V-U} of $V - U$, and then the mean $E(V - U)$ by integrating the density. Finally, check that your answers to (a), (b), (c) satisfy $E(V - U) = E(V) - E(U)$.

2.7. Let S and T be exponentially distributed with rates λ and μ. Let $U = \min\{S, T\}$ and $V = \max\{S, T\}$. Find (a) EU. (b) $E(V - U)$, (c) EV. (d) Use the identity $V = S + T - U$ to get a different looking formula for EV and verify the two are equal.

Part (a)

- $\mathbb{P}(U > x) = \mathbb{P}(S, T > x) = \mathbb{P}(S > x)\mathbb{P}(T > x) = e^{-(\lambda+\mu)x} \Rightarrow U \sim \text{Exp}(\lambda + \mu) \Rightarrow \mathbb{E}[U] = \frac{1}{\lambda + \mu}$

Part (b)

- Claim: $\mathbb{P}(X - Y > s) = \frac{\lambda_2}{\lambda_1 + \lambda_2} e^{-\lambda_1 s}$ for $X \sim \text{Exp}(\lambda_1), Y \sim \text{Exp}(\lambda_2)$ and $s > 0$
  - $\mathbb{P}(X > Y + s) = \int_0^\infty f_Y(y)\mathbb{P}(X > y + s)dy = \int_0^\infty \lambda_2 e^{-\lambda_2 y} \cdot e^{-\lambda_1(y+s)} dy = \frac{\lambda_2}{\lambda_1 + \lambda_2} e^{-\lambda_1 s}$
- $\mathbb{P}(V - U > x) = \mathbb{P}(T - S > x) + \mathbb{P}(S - T > x) = \frac{\lambda}{\lambda + \mu} e^{-\mu x} + \frac{\mu}{\lambda + \mu} e^{-\lambda x}$, for $x > 0$
- $f_{V-U}(x) = -\frac{d}{dx}\left(\frac{\lambda}{\lambda + \mu} e^{-\mu x} + \frac{\mu}{\lambda + \mu} e^{-\lambda x}\right) = \frac{\lambda\mu}{\lambda + \mu}\left(e^{-\mu x} + e^{-\lambda x}\right)$, for $x > 0$
- $\mathbb{E}[V - U] = \int_0^\infty x \cdot f_{V-U}(x)dx = \int_0^\infty x \frac{\lambda\mu}{\lambda + \mu}\left(e^{-\mu x} + e^{-\lambda x}\right)dx = \frac{\lambda\mu}{\lambda + \mu}\left(\frac{1}{\mu^2} + \frac{1}{\lambda^2}\right)$

Part (c)

- $\mathbb{P}(V < x) = \mathbb{P}(S, T < x) = \mathbb{P}(S < x)\mathbb{P}(T < x) = \left(1 - e^{-x\lambda}\right)\left(1 - e^{-x\mu}\right)$

1 Durrett 2.4 **10 / 10**

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- 3 pts Minor errors. Examples: Incorrectly evaluated integrals, arithmetic or algebra mistakes

- 5 pts Substantial errors. Examples: integrals set up incorrectly

- 8 pts Serious errors. Examples: does not set up any appropriate integral, does not use exponential density at all

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- $\mathbb{P}(T_2 + t < T_1) = \int_0^\infty f_{T_2}(s)\mathbb{P}(T_1 > s + t)ds = \int_0^\infty \lambda_2 e^{-\lambda_2 s} \cdot e^{-\lambda_1(s+t)}\, ds = \frac{\lambda_2}{\lambda_1 + \lambda_2} e^{-\lambda_1 t}$

Question 2

Durrett Exercise 2.7 (a), (b), (c), page 93. (Ignore part (d).) In part (b), compute first $P(V - U > s)$ for $s > 0$ either by integrating densities of S and T or by conditioning on the events $S < T$ and $T < S$. From $P(V - U > s)$ deduce the density function f_{V-U} of $V - U$, and then the mean $E(V - U)$ by integrating the density. Finally, check that your answers to (a), (b), (c) satisfy $E(V - U) = E(V) - E(U)$.

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Part (b)

- Claim: $\mathbb{P}(X - Y > s) = \frac{\lambda_2}{\lambda_1 + \lambda_2} e^{-\lambda_1 s}$ for $X \sim \text{Exp}(\lambda_1), Y \sim \text{Exp}(\lambda_2)$ and $s > 0$
  - $\mathbb{P}(X > Y + s) = \int_0^\infty f_Y(y)\mathbb{P}(X > y + s)dy = \int_0^\infty \lambda_2 e^{-\lambda_2 y} \cdot e^{-\lambda_1(y+s)} dy = \frac{\lambda_2}{\lambda_1 + \lambda_2} e^{-\lambda_1 s}$
- $\mathbb{P}(V - U > x) = \mathbb{P}(T - S > x) + \mathbb{P}(S - T > x) = \frac{\lambda}{\lambda + \mu} e^{-\mu x} + \frac{\mu}{\lambda + \mu} e^{-\lambda x}$, for $x > 0$
- $f_{V-U}(x) = -\frac{d}{dx}\left(\frac{\lambda}{\lambda + \mu} e^{-\mu x} + \frac{\mu}{\lambda + \mu} e^{-\lambda x}\right) = \frac{\lambda\mu}{\lambda + \mu}(e^{-\mu x} + e^{-\lambda x})$, for $x > 0$
- $\mathbb{E}[V - U] = \int_0^\infty x \cdot f_{V-U}(x)dx = \int_0^\infty x \frac{\lambda\mu}{\lambda + \mu}(e^{-\mu x} + e^{-\lambda x})dx = \frac{\lambda\mu}{\lambda + \mu}\left(\frac{1}{\mu^2} + \frac{1}{\lambda^2}\right)$

Part (c)

- $\mathbb{P}(V < x) = \mathbb{P}(S, T < x) = \mathbb{P}(S < x)\mathbb{P}(T < x) = (1 - e^{-x\lambda})(1 - e^{-x\mu})$

- $f_V(x) = \frac{d}{dx}\left[(1-e^{-x\lambda})(1-e^{-x\mu})\right] = \lambda e^{-x\lambda} + \mu e^{-x\mu} - (\lambda+\mu)e^{-x(\lambda+\mu)}$
- $\mathbb{E}[V] = \int_0^\infty x \cdot f_V(x)dx = \int_0^\infty x\left(\lambda e^{-x\lambda} + \mu e^{-x\mu} - (\lambda+\mu)e^{-x(\lambda+\mu)}\right)dx = \frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\lambda+\mu}$
- Verify: $\mathbb{E}[V] - \mathbb{E}[U] = \left(\frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\lambda+\mu}\right) - \frac{1}{\lambda+\mu} = \frac{\lambda_2 + \mu^2}{\lambda\mu(\lambda+\mu)} = \frac{\lambda\mu}{\lambda+\mu}\left(\frac{1}{\mu^2} + \frac{1}{\lambda^2}\right) = \mathbb{E}[V-U]$

Question 3

Durrett Exercise 2.9 (b), page 93. (Ignore part (a).) For systematic notation, let A_i denote the amount of time Al spends at server i, and B_i the amount of time Bob spends at server i, for $i = 1, 2$.

Hint. Draw a picture of the time line to understand how Al and Bob move through the servers. Use your answer from 2.7. This problem requires no integration.

2.9. In a hardware store you must first go to server 1 to get your goods and then go to a server 2 to pay for them. Suppose that the times for the two activities are exponentially distributed with means 6 minutes and 3 minutes. (a) Compute the average amount of time it take Bob to get his goods and pay if when he comes in there is one customer named Al with server 1 and no one at server 2. (b) Find the answer when times for the two activities are exponentially distributed with rates λ and μ.

- Let $A_1, B_1 \sim \text{Exp}(\lambda)$ and $A_2, B_2 \sim \text{Exp}(\mu)$ be mutally independent
- If $B_1 \le A_2$, then the total waiting time is $A_1 + A_2 + B_2$
- If $B_1 > A_2$, then the total waiting time is $A_1 + B_1 + B_2$
- Therefore, $\mathbb{E}[A_1 + \max\{B_1, A_2\} + B_2] = \frac{1}{\lambda} + \left(\frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\lambda+\mu}\right) + \frac{1}{\mu} = \frac{2}{\lambda} + \frac{2}{\mu} - \frac{1}{\lambda+\mu}$

Question 4

Durrett Exercise 2.13, page 93.

Hint. Let N_i denote a Poisson process of rate λ_i. For part (a), express the event in terms of N_1, N_2 and N_3. You may assume that the shocks come indepently, so N_1, N_2 and N_3 are independent. Part (b) follows quickly from (a). For part (c) check whether $P(U > s, V > t)$ equals $P(U > s) \cdot P(V > t)$.

2.13. A machine has two critically important parts and is subject to three different types of shocks. Shocks of type i occur at times of a Poisson process with rate λ_i. Shocks of types 1 break part 1, those of type 2 break part 2, while those of type 3 break both parts. Let U and V be the failure times of the two parts. (a) Find $P(U > s, V > t)$. (b) Find the distribution of U and the distribution of V. (c) Are U and V independent?

Part (a)

- $\mathbb{P}(U > s, V > t) = \mathbb{P}\left(N_1(s) = 0, N_2(t) = 0, N_3(\max\{s,t\}) = 0\right) = e^{-\lambda_1 s}e^{-\lambda_2 t}e^{-\lambda_3 \max\{s,t\}}$

Part (b)

- $\mathbb{P}(U > s) = \mathbb{P}\left(N_1(s) = 0, N_3(s) = 0\right) = e^{-\lambda_1 s}e^{-\lambda_3 s} = e^{-(\lambda_1+\lambda_3)s} \Rightarrow f_U(s) = 1 - e^{-(\lambda_1+\lambda_3)s}$
- $\mathbb{P}(V > t) = \mathbb{P}\left(N_2(t) = 0, N_3(t) = 0\right) = e^{-\lambda_2 t}e^{-\lambda_3 t} = e^{-(\lambda_2+\lambda_3)t} \Rightarrow f_V(t) = 1 - e^{-(\lambda_2+\lambda_3)t}$

2 Durrett 2.7abc 10 / 10

✓ **- 0 pts** **Correct**

- 1 pts (a) Minor error. Examples: arithmetic or algebra

- 3 pts (a) Little progress to solution. Examples: did not use properties of exponential distribution in any way

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- 4 pts (b) Little progress to solution. Examples: did not use properties of exponential distribution in any way

- 1 pts (c) Minor errors. Examples: computational mistakes

- 3 pts (c) Little progress to solution. Examples: did not use properties of exponential distribution in any way

- 10 pts No work submitted

- $f_V(x) = \frac{d}{dx}\left[(1-e^{-x\lambda})(1-e^{-x\mu})\right] = \lambda e^{-x\lambda} + \mu e^{-x\mu} - (\lambda+\mu)e^{-x(\lambda+\mu)}$
- $\mathbb{E}[V] = \int_0^\infty x \cdot f_V(x)dx = \int_0^\infty x\left(\lambda e^{-x\lambda} + \mu e^{-x\mu} - (\lambda+\mu)e^{-x(\lambda+\mu)}\right)dx = \frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\lambda+\mu}$
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Question 3

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Question 4

Durrett Exercise 2.13, page 93.

Hint. Let N_i denote a Poisson process of rate λ_i. For part (a), express the event in terms of N_1, N_2 and N_3. You may assume that the shocks come indepently, so N_1, N_2 and N_3 are independent. Part (b) follows quickly from (a). For part (c) check whether $P(U > s, V > t)$ equals $P(U > s) \cdot P(V > t)$.

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Part (b)

- $\mathbb{P}(U > s) = \mathbb{P}\left(N_1(s) = 0, N_3(s) = 0\right) = e^{-\lambda_1 s}e^{-\lambda_3 s} = e^{-(\lambda_1+\lambda_3)s} \Rightarrow f_U(s) = 1 - e^{-(\lambda_1+\lambda_3)s}$
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3 Durrett 2.9b **10 / 10**

- ✓ **- 0 pts** **Correct**
- **- 3 pts** Minor errors. Examples: arithmetic or algebra errors
- **- 5 pts** Substantial errors. Examples: incorrect expression for Bob's total waiting time
- **- 8 pts** Little progress is made towards the solution
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- $f_V(x) = \frac{d}{dx}\left[(1-e^{-x\lambda})(1-e^{-x\mu})\right] = \lambda e^{-x\lambda} + \mu e^{-x\mu} - (\lambda+\mu)e^{-x(\lambda+\mu)}$
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Part (b)

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Part (c)

- $\mathbb{P}(U > s)\mathbb{P}(V > t) = e^{-(\lambda_1+\lambda_3)s}e^{-(\lambda_2+\lambda_3)t} \neq e^{-\lambda_1 s}e^{-\lambda_2 t}e^{-\lambda_3 \max\{s,t\}} = \mathbb{P}(U > s, V > t)$
- Therefore U and V are not independent

Question 5

Consider a Poisson process of rate λ and let s be a fixed positive number. Let σ be the random amount of time from s till the next arrival. In symbols,

$$\sigma = \left(\min_{k:T_k>s} T_k\right) - s.$$

Calculate rigorously the probability $P(\sigma > a)$ for real $a > 0$ by using the density functions of the arrival times T_k and the interarrival times τ_k.

Hint. Draw a picture of the time line. and express the event $\{\sigma > a\}$ in terms of the arrival times $\{T_k\}$ whose density functions we know. Since $s + \sigma$ is one of the arrival times, you can decompose the probability $P(\sigma > a)$ into different cases according to which T_k is equal to $s + \sigma$.

- $\mathbb{P}(\sigma > a) = \mathbb{P}(s + \sigma > s + a) = \sum_{k=0}^{\infty} \mathbb{P}(T_k < s < a + s < T_{k+1})$

$$= \sum_{k=0}^{\infty} \mathbb{P}(\tau_{k+1} > a + s - T_k, T_k < s) = \sum_{k=0}^{\infty}\left(\int_0^s f_{T_k}(t)\mathbb{P}(\tau_{k+1} > a + s - t)dt\right)$$

$$= \sum_{k=0}^{\infty}\left(\int_0^s \left(\lambda e^{-\lambda t}\frac{(\lambda t)^{k-1}}{(k-1)!}\right)\left(e^{-\lambda(a+s-t)}\right)dt\right) = \sum_{k=0}^{\infty}\left(e^{-(a+s)\lambda}\lambda^k \int_0^s \frac{t^{k-1}}{(k-1)!}dt\right)$$

$$= \sum_{k=0}^{\infty}\left(e^{-(a+s)\lambda}\lambda^k\left[\frac{t^k}{k!}\right]_{t=0}^{t=s}\right) = e^{-\lambda(a+s)}\sum_{k=0}^{\infty}\frac{(\lambda s)^k}{k!} = e^{-\lambda(a+s)} \cdot e^{\lambda s} = e^{-\lambda a}$$

4 Durrett 2.13 10 / 10

✓ **- 0 pts** **Correct**

- 1 pts (a) Minor computational error

- 2 pts (a) Substantial error. Examples: incorrectly set up probability in terms of the Poisson processes

- 4 pts (a) Little progress to the solution

- 1 pts (b) Minor computational error

- 3 pts (b) Substantial error

- 1 pts (c) Incorrect conclusion due to minor error

- 3 pts (c) Serious errors: Incorrect definition of independence

- 10 pts No work submitted

Part (c)

- $\mathbb{P}(U > s)\mathbb{P}(V > t) = e^{-(\lambda_1+\lambda_3)s}e^{-(\lambda_2+\lambda_3)t} \neq e^{-\lambda_1 s}e^{-\lambda_2 t}e^{-\lambda_3 \max\{s,t\}} = \mathbb{P}(U > s, V > t)$
- Therefore U and V are not independent

Question 5

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- $\mathbb{P}(\sigma > a) = \mathbb{P}(s + \sigma > s + a) = \sum_{k=0}^{\infty} \mathbb{P}(T_k < s < a + s < T_{k+1})$

$$= \sum_{k=0}^{\infty} \mathbb{P}(\tau_{k+1} > a + s - T_k, T_k < s) = \sum_{k=0}^{\infty}\left(\int_0^s f_{T_k}(t)\mathbb{P}(\tau_{k+1} > a + s - t)dt\right)$$

$$= \sum_{k=0}^{\infty}\left(\int_0^s \left(\lambda e^{-\lambda t}\frac{(\lambda t)^{k-1}}{(k-1)!}\right)\left(e^{-\lambda(a+s-t)}\right)dt\right) = \sum_{k=0}^{\infty}\left(e^{-(a+s)\lambda}\lambda^k \int_0^s \frac{t^{k-1}}{(k-1)!}dt\right)$$

$$= \sum_{k=0}^{\infty}\left(e^{-(a+s)\lambda}\lambda^k \left[\frac{t^k}{k!}\right]_{t=0}^{t=s}\right) = e^{-\lambda(a+s)}\sum_{k=0}^{\infty}\frac{(\lambda s)^k}{k!} = e^{-\lambda(a+s)} \cdot e^{\lambda s} = e^{-\lambda a}$$

5 Problem 5 **10 / 10**

- ✓ **- 0 pts** **Correct**
- **- 3 pts** Computational errors
- **- 5 pts** Substantial error. Examples: Mistake in integral set up
- **- 8 pts** Little progress to the solution
- **- 10 pts** No work submitted
- **- 3 pts** Forgot the summation