Math 632 004 HW06

Shawn Zhong

TOTAL POINTS

67 / 70

QUESTION 1

1 Problem 1 **10 / 10**

✓ **- 0 pts** **All correct**

QUESTION 2

2 Durrett 2.36 **10 / 10**

✓ **- 0 pts** **Correct**

QUESTION 3

3 Durrett 2.38 **10 / 10**

✓ **- 0 pts** **Correct**

QUESTION 4

4 Durrett 2.44 **10 / 10**

✓ **- 0 pts** **Correct**

QUESTION 5

5 Durrett 2.48 **10 / 10**

✓ **- 0 pts** **Correct**

QUESTION 6

6 Durrett 2.49 **7 / 10**

✓ **- 3 pts** **(a) Conceptual errors. Examples: incorrect value for expectation or variance is used**

QUESTION 7

7 Durrett 2.51 **10 / 10**

✓ **- 0 pts** **Correct**

HW6 - Problem & Solution

Saturday, November 3, 2018 8:10 AM

Math 632 Lecture 4, Fall 2018, Homework 6

Due Tuesday November 13 by 10:05am

Question 1

Let $\{N(t) : t \geq 0\}$ be a rate λ Poisson process, $\{T_k\}_{k\geq 1}$ the arrival times of the process, $T_0 = 0$, and $\tau_k = T_k - T_{k-1}$ for $k \geq 1$ the interarrival times. Calculate the probabilities below. When unspecified nonnegative integers j and/or k appear in the question, your answer should cover all possible cases. When units are used, suppose the time unit is second.

(a) $P(N(2) = j, N(5) = k)$.

(b)

$$P(\text{after the 3rd arrival there are no arrivals for 20 seconds, but after that the next two arrivals come within 10 seconds}).$$

(c) $P(N(3) = k \,|\, T_2 \leq 3)$.

(d) $P(N(2) = k \,|\, T_3 > 4)$.

(e) $P(T_2 \leq 3 \,|\, N(4) = 5)$. Explain how your answer can be expressed in terms of a certain binomial probability mass function.

Part (a)

- $\mathbb{P}\big(N(2) = j, N(5) = k\big) = \mathbb{P}\big(N(0,2] = j, N(2,5] = k - j\big) = \mathbb{P}\big(N(0,2] = j\big)\mathbb{P}\big(N(2,5] = k - j\big)$
- For $k \geq j \geq 0$, $\mathbb{P}\big(N(2) = j, N(5) = k\big) = \left(e^{-2\lambda}\frac{(2\lambda)^j}{j!}\right)\left(e^{-3\lambda}\frac{(3\lambda)^{k-j}}{(k-j)!}\right) = e^{-5\lambda}\frac{(3\lambda)^{k-j}(2\lambda)^j}{j!\,(k-j)!}$
- Otherwise, $\mathbb{P}\big(N(2) = j, N(5) = k\big) = 0$

Part (b)

- $\mathbb{P}(N(t, t+20] = 0, N(t+20, t+30] \geq 2)$, where t is the third arrival time

$$= \big(\mathbb{P}(N(0,20] = 0)\big)\big(1 - \mathbb{P}(N(20,30] = 0) - \mathbb{P}(N(20,30] = 1)\big)$$

$$= \left(e^{-20\lambda}\frac{(20\lambda)^0}{0!}\right)\left(1 - e^{-10\lambda}\frac{(10\lambda)^0}{0!} - e^{-10\lambda}\frac{(10\lambda)^1}{1!}\right)$$

$$= e^{-20\lambda}\big(1 - e^{-10\lambda} - 10\lambda \cdot e^{-10\lambda}\big)$$

Part (c)

- $\mathbb{P}\big(N(3) = k \big| T_2 \leq 3\big) = \mathbb{P}(N(3) = k | N(3) \geq 2) = \dfrac{\mathbb{P}(N(3) = k, N(3) \geq 2)}{\mathbb{P}(N(3) \geq 2)}$
- For $k < 2$, $\mathbb{P}\big(N(3) = k \big| T_2 \leq 3\big) = 0$

- Otherwise, $\mathbb{P}(N(3)=k|N(3)\geq 2)=\frac{\mathbb{P}(N(3)=k)}{1-\mathbb{P}(N(3)=1)-\mathbb{P}(N(3)=0)}=\frac{e^{-3\lambda}\frac{(3\lambda)^k}{k!}}{1-e^{-3\lambda}-e^{-3\lambda}3\lambda}$

Part (d)

- For $k\geq 3, \mathbb{P}\big(N(2)=k\big|T_3>4\big)=0$. Assume otherwise.
- $\mathbb{P}\big(N(2)=k\big|T_3>4\big)=\mathbb{P}(N(2)=k|N(4)<3)$

$$=\frac{\mathbb{P}(N(2)=k,N(4)<3)}{\mathbb{P}(N(4)<3)}$$

$$=\frac{\mathbb{P}(N(2)=k)\mathbb{P}(N(2,4]\leq 2-k)}{\mathbb{P}(N(4)\leq 2)}$$

$$=\frac{\mathbb{P}(N(2)=k)}{\mathbb{P}(N(4)=2)+\mathbb{P}(N(4)=1)+\mathbb{P}(N(4)=0)}\sum_{i=0}^{2-k}\mathbb{P}(N(2,4]=i)$$

$$=\frac{e^{-2\lambda}\frac{(2\lambda)^k}{k!}}{e^{-4\lambda}\frac{(2\lambda)^2}{2!}+e^{-4\lambda}4\lambda+e^{-4\lambda}}\sum_{i=0}^{2-k}e^{-2\lambda}\frac{(2\lambda)^i}{i!}$$

$$=\frac{1}{8\lambda^2+4\lambda+1}\cdot\frac{(2\lambda)^k}{k!}\cdot\sum_{i=0}^{2-k}\frac{(2\lambda)^i}{i!}$$

- For $k=2, \mathbb{P}\big(N(2)=k\big|T_3>4\big)=\frac{1}{8\lambda^2+4\lambda+1}\cdot\frac{4\lambda^2}{2!}=\frac{2\lambda^2}{8\lambda^2+4\lambda+1}$
- For $k=1, \mathbb{P}\big(N(2)=k\big|T_3>4\big)=\frac{1}{8\lambda^2+4\lambda+1}\cdot 2\lambda\cdot(1+2\lambda)=\frac{4\lambda^2+2\lambda}{8\lambda^2+4\lambda+1}$
- For $k=0, \mathbb{P}\big(N(2)=k\big|T_3>4\big)=\frac{1}{8\lambda^2+4\lambda+1}\cdot 1\cdot\left(1+2\lambda+\frac{(2\lambda)^2}{2!}\right)=\frac{2\lambda^2+2\lambda+1}{8\lambda^2+4\lambda+1}$
- Therefore $\mathbb{P}\big(N(2)=k\big|T_3>4\big)=\begin{cases}\frac{2\lambda^2+2\lambda+1}{8\lambda^2+4\lambda+1} & k=0\\ \frac{4\lambda^2+2\lambda}{8\lambda^2+4\lambda+1} & k=1\\ \frac{2\lambda^2}{8\lambda^2+4\lambda+1} & k=2\\ 0 & k\geq 3\end{cases}$

Part (e)

- $\mathbb{P}\big(T_2\leq 3\big|N(4)=5\big)=\mathbb{P}(N(3)\geq 2|N(4)=5)=\frac{\mathbb{P}(N(3)\geq 2,N(4)=5)}{\mathbb{P}(N(4)=5)}$

$$=\sum_{i=2}^{5}\frac{\mathbb{P}(N(3)=i,N(3,4]=5-i)}{\mathbb{P}(N(4)=5)}=\sum_{i=2}^{5}\frac{e^{-3\lambda}\frac{(3\lambda)^i}{i!}\cdot e^{-\lambda}\frac{\lambda^{5-i}}{(5-i)!}}{e^{-4\lambda}\frac{(4\lambda)^5}{5!}}=\sum_{i=2}^{5}\binom{5}{i}\left(\frac{3}{4}\right)^i\left(\frac{1}{4}\right)^{5-i}$$

- Let $X\sim\text{Bin}(5,3/4)$, then $\mathbb{P}\big(T_2\leq 3\big|N(4)=5\big)=\sum_{i=2}^{5}p_X(i)$

Question 2

1 Problem 1 **10 / 10**

✓ **- 0 pts** **All correct**

2.36. Customers arrive at an automated teller machine at the times of a Poisson process with rate of 10 per hour. Suppose that the amount of money withdrawn on each transaction has a mean of \$30 and a standard deviation of \$20. Find the mean and standard deviation of the total withdrawals in 8 hours.

- Given $\mathbb{E}[Y_1] = 30, \mathrm{Var}[Y_1] = 20^2 = 400, \lambda = 10$
- $\mathbb{E}[S(8)] = \lambda \cdot 8 \cdot \mathbb{E}[Y_1] = 10 \cdot 8 \cdot 30 = 2{,}400$
- $\mathrm{Var}[S(8)] = \lambda \cdot 8 \cdot \left(\mathrm{Var}[Y_1] + \left(\mathbb{E}[Y_1]\right)^2\right) = 10 \cdot 8 \cdot (400 + 30^2) = 104000$
- $\mathrm{SD}[S(8)] = \sqrt{\mathrm{Var}[S(8)]} \approx 17.96$

Question 3

2.38. Let S_t be the price of stock at time t and suppose that at times of a Poisson process with rate λ the price is multiplied by a random variable $X_i > 0$ with mean μ and variance σ^2. That is,

$$S_t = S_0 \prod_{i=1}^{N(t)} X_i$$

where the product is 1 if $N(t) = 0$. Find $ES(t)$ and $\mathrm{var}\, S(t)$.

Part (a)

- $\mathbb{E}[S(t)|N(t) = n] = \mathbb{E}\left[S_0 \prod_{i=1}^{n} X_i \middle| N(t) = n\right] = \mathbb{E}[S_0] \prod_{i=1}^{n} \mathbb{E}[X_i|N(t) = n] = \mathbb{E}[S_0]\mu^n$
- $\mathbb{E}[S(t)] = \mathbb{E}[\mathbb{E}[S(t)|N(t)]] = \mathbb{E}[\mathbb{E}[S_0]\mu^{N(t)}] = \mathbb{E}[S_0]\mathbb{E}[\mu^{N(t)}]$

$$= \mathbb{E}[S_0] \sum_{k=0}^{\infty} \mu^k \mathbb{P}(N(t) = k) = \mathbb{E}[S_0] \sum_{k=0}^{\infty} \mu^k e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

$$= \mathbb{E}[S_0] e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\mu \lambda t)^k}{k!} = \mathbb{E}[S_0] e^{-\lambda t} e^{\mu \lambda t} = \mathbb{E}[S_0] e^{(\mu - 1)\lambda t}$$

Part (b)

- $\mathbb{E}\left[(S(t))^2 \middle| N(t) = n\right] = \mathbb{E}\left[S_0^2 \prod_{i=1}^{n} X_i^2 \middle| N(t) = n\right] = \mathbb{E}[S_0^2] \prod_{i=1}^{n} \mathbb{E}\left[X_i^2 | N(t) = n\right]$

$$= \mathbb{E}[S_0^2] \prod_{i=1}^{n} \mathbb{E}\left[X_i^2\right] = \mathbb{E}[S_0^2] \prod_{i=1}^{n} \left(\left(\mathbb{E}[X_i]\right)^2 + \mathrm{Var}(x)\right) = \mathbb{E}[S_0^2](\mu^2 + \sigma^2)^n$$

- $\mathbb{E}\left[(S(t))^2\right] = \mathbb{E}\left[\mathbb{E}\left[(S(t))^2 \middle| N(t)\right]\right] = \mathbb{E}\left[\mathbb{E}[S_0^2](\mu^2 + \sigma^2)^{N(t)}\right]$

$$= \mathbb{E}[S_0^2]\mathbb{E}\left[(\mu^2 + \sigma^2)^{N(t)}\right] = \mathbb{E}[S_0^2] \sum_{k=0}^{\infty} (\mu^2 + \sigma^2)^k \mathbb{P}(N(t) = k)$$

$$= \mathbb{E}[S_0^2] \sum_{k=0}^{\infty} (\mu^2 + \sigma^2)^k e^{-\lambda t} \frac{(\lambda t)^k}{k!} = \mathbb{E}[S_0^2] e^{-\lambda t} \sum_{k=0}^{\infty} \frac{[(\mu^2 + \sigma^2)\lambda t]^k}{k!}$$

2 Durrett 2.36 10 / 10

✓ - 0 pts Correct

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- $\mathbb{E}[S(t)] = \mathbb{E}[\mathbb{E}[S(t)|N(t)]] = \mathbb{E}[\mathbb{E}[S_0]\mu^{N(t)}] = \mathbb{E}[S_0]\mathbb{E}[\mu^{N(t)}]$

$$= \mathbb{E}[S_0] \sum_{k=0}^{\infty} \mu^k \mathbb{P}(N(t) = k) = \mathbb{E}[S_0] \sum_{k=0}^{\infty} \mu^k e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

$$= \mathbb{E}[S_0] e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\mu \lambda t)^k}{k!} = \mathbb{E}[S_0] e^{-\lambda t} e^{\mu \lambda t} = \mathbb{E}[S_0] e^{(\mu - 1)\lambda t}$$

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- $\mathbb{E}\left[(S(t))^2 \middle| N(t) = n\right] = \mathbb{E}\left[S_0^2 \prod_{i=1}^{n} X_i^2 \middle| N(t) = n\right] = \mathbb{E}[S_0^2] \prod_{i=1}^{n} \mathbb{E}\left[X_i^2 | N(t) = n\right]$

$$= \mathbb{E}[S_0^2] \prod_{i=1}^{n} \mathbb{E}\left[X_i^2\right] = \mathbb{E}[S_0^2] \prod_{i=1}^{n} \left((\mathbb{E}[X_i])^2 + \mathrm{Var}(x)\right) = \mathbb{E}[S_0^2](\mu^2 + \sigma^2)^n$$

- $\mathbb{E}\left[(S(t))^2\right] = \mathbb{E}\left[\mathbb{E}\left[(S(t))^2 \middle| N(t)\right]\right] = \mathbb{E}\left[\mathbb{E}[S_0^2](\mu^2 + \sigma^2)^{N(t)}\right]$

$$= \mathbb{E}[S_0^2]\mathbb{E}\left[(\mu^2 + \sigma^2)^{N(t)}\right] = \mathbb{E}[S_0^2] \sum_{k=0}^{\infty} (\mu^2 + \sigma^2)^k \, \mathbb{P}(N(t) = k)$$

$$= \mathbb{E}[S_0^2] \sum_{k=0}^{\infty} (\mu^2 + \sigma^2)^k e^{-\lambda t} \frac{(\lambda t)^k}{k!} = \mathbb{E}[S_0^2] e^{-\lambda t} \sum_{k=0}^{\infty} \frac{[(\mu^2 + \sigma^2)\lambda t]^k}{k!}$$

$$= \mathbb{E}[S_0^2]e^{-\lambda t}e^{(\mu^2+\sigma^2)\lambda t} = \mathbb{E}[S_0^2]e^{(\mu^2+\sigma^2-1)\lambda t}$$

- $\text{Var}[S(t)] = \mathbb{E}\left[(S(t))^2\right] - (\mathbb{E}[S(t)])^2 = \mathbb{E}[S_0^2]e^{(\mu^2+\sigma^2-1)\lambda t} - (\mathbb{E}[S_0])^2 e^{2(\mu-1)\lambda t}$

Question 4

2.44. Ellen catches fish at times of a Poisson process with rate 2 per hour. 40% of the fish are salmon, while 60% of the fish are trout. What is the probability she will catch exactly 1 salmon and 2 trout if she fishes for 2.5 hours?

- $N(t) :=$ # fish Ellen have caught up to time $t \Rightarrow N(t)$ is a P.P with $\lambda = 2$
- $N_s(t) :=$ # salmon Ellen have caught up to time $t \Rightarrow N_s(t)$ is a P.P with $\lambda_s = 40\%\lambda = 0.8$
- $N_t(t) :=$ # trout Ellen have caught up to time $t \Rightarrow N_t(t)$ is a P.P with $\lambda_t = 60\%\lambda = 1.2$
- $\mathbb{P}(N_s(2.5) = 1, N_t(2.5) = 2) = \left[e^{-2.5\lambda_s}\frac{(2.5\lambda_s)^1}{1!}\right]\left[e^{-2.5\lambda_t}\frac{(2.5\lambda_t)^2}{2!}\right] = 9e^{-5} = 0.06064$

Question 5

2.48. When a power surge occurs on an electrical line, it can damage a computer without a surge protector. There are three types of surges: "small" surges occur at rate 8 per day and damage a computer with probability 0.001; "medium" surges occur at rate 1 per day and will damage a computer with probability 0.01; "large" surges occur at rate 1 per month and damage a computer with probability 0.1. Assume that months are 30 days. (a) what is the expected number of power surges per month? (b) What is the expected number of computer damaging power surges per month? (c) What is the probability a computer will not be damaged in one month? (d) What is the probability that the first computer damaging surge is a small one?

Part (a)

- $N_s(t) :=$ the number of small surges up to day t, then $N_s(t)$ is a P.P. with $\lambda_s = 8 \times 30 = 240$
- $N_m(t) :=$ the number of medium surges up to day t, then $N_m(t)$ is a P.P. with $\lambda_m = 1 \times 30 = 30$
- $N_l(t) :=$ the number of large surges up to day t, then $N_l(t)$ is a P.P. with $\lambda_l = 1$
- $\mathbb{E}[N_s + N_m + N_l] = \lambda_s + \lambda_m + \lambda_l = 240 + 30 + 1 = 271$

Part (b)

- $N_s'(t) :=$ # small surges with damage up to day $t \Rightarrow N_s(t)$ is a P.P. with $\lambda_s' = 0.001\lambda_s = 0.24$
- $N_m'(t) :=$ # medium surges with damage up to day $t \Rightarrow N_m(t)$ is a P.P. with $\lambda_m' = 0.01\lambda_m = 0.3$
- $N_l'(t) :=$ # large surges with damage up to day $t \Rightarrow N_l(t)$ is a P.P. with $\lambda_l' = 0.1\lambda_l = 0.1$
- $\mathbb{E}[N_s' + N_m' + N_l'] = \lambda_s' + \lambda_m' + \lambda_l' = 0.64$

Part (c)

- Define $N'(t) := N_s'(t) + N_m'(t) + N_l'(t)$, then $N'(t)$ is a P.P. with $\lambda' = \lambda_s' + \lambda_m' + \lambda_l' = 0.64$
- $\mathbb{P}(N'(1) = 0) = e^{-\lambda'} = e^{-0.64} \approx 0.527292$

Part (d)

- Method 1

3 Durrett 2.38 **10 / 10**

✓ **- 0 pts** Correct

$$= \mathbb{E}[S_0^2]e^{-\lambda t}e^{(\mu^2+\sigma^2)\lambda t} = \mathbb{E}[S_0^2]e^{(\mu^2+\sigma^2-1)\lambda t}$$

- $\text{Var}[S(t)] = \mathbb{E}\left[(S(t))^2\right] - (\mathbb{E}[S(t)])^2 = \mathbb{E}[S_0^2]e^{(\mu^2+\sigma^2-1)\lambda t} - \left(\mathbb{E}[S_0]\right)^2 e^{2(\mu-1)\lambda t}$

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2.44. Ellen catches fish at times of a Poisson process with rate 2 per hour. 40% of the fish are salmon, while 60% of the fish are trout. What is the probability she will catch exactly 1 salmon and 2 trout if she fishes for 2.5 hours?

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- Define $N'(t) \coloneqq N_s'(t) + N_m'(t) + N_l'(t)$, then $N'(t)$ is a P.P. with $\lambda' = \lambda_s' + \lambda_m' + \lambda_l' = 0.64$
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4 Durrett 2.44 **10 / 10**

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$= \mathbb{E}[S_0^2]e^{-\lambda t}e^{(\mu^2+\sigma^2)\lambda t} = \mathbb{E}[S_0^2]e^{(\mu^2+\sigma^2-1)\lambda t}$

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- $\mathbb{E}[N_s' + N_m' + N_l'] = \lambda_s' + \lambda_m' + \lambda_l' = 0.64$

Part (c)

- Define $N'(t) := N_s'(t) + N_m'(t) + N_l'(t)$, then $N'(t)$ is a P.P. with $\lambda' = \lambda_s' + \lambda_m' + \lambda_l' = 0.64$
- $\mathbb{P}(N'(1) = 0) = e^{-\lambda'} = e^{-0.64} \approx 0.527292$

Part (d)

- Method 1

- $\dfrac{\lambda_s}{\lambda_s + \lambda_m + \lambda_l} = \dfrac{0.24}{0.24 + 0.3 + 0.1} = \dfrac{3}{8}$
- Method 2
  - Define $T_s \sim \text{Exp}(\lambda_s), T_m \sim \text{Exp}(\lambda_m), T_l \sim \text{Exp}(\lambda_l)$
  - $\mathbb{P}(T_s < T_m, T_l) = \mathbb{P}(T_s < T_m < T_l) + \mathbb{P}(T_s < T_l < T_m)$

$$= \int_0^\infty f_{T_l}(l) \int_0^l f_{T_m}(m) \int_0^m f_{T_s}(s) ds\, dm\, dl + \int_0^\infty f_{T_m}(m) \int_0^m f_{T_l}(l) \int_0^l f_{T_s}(s) ds\, dl\, dm$$

$$= \frac{9}{32} + \frac{3}{32} = \frac{3}{8}$$

Question 6

2.49. Wayne Gretsky scored a Poisson mean 6 number of points per game. 60% of these were goals and 40% were assists (each is worth one point). Suppose he is paid a bonus of 3K for a goal and 1K for an assist. (a) Find the mean and standard deviation for the total revenue he earns per game. (b) What is the probability that he has 4 goals and 2 assists in one game? (c) Conditional on the fact that he had 6 points in a game, what is the probability he had 4 in the first half?

Part (a)

- Assume a game is a continuous process (*i.e.* game 1.5 means half time of second game)
- $N(t) :=$ # points up to game $t \Rightarrow N(t)$ is a Poisson Process with $\lambda = 6$
- $N_g(t) :=$ # goals up to game $t \Rightarrow N_g(t)$ is a Poisson Process with $\lambda_g = 0.6\lambda = 3.6$
- $N_a(t) :=$ # assists up to game $t \Rightarrow N_a(t)$ is a Poisson Process with $\lambda_a = 0.4\lambda = 2.4$
- $N_g'(t) :=$ # bonus from goals up to game $t \Rightarrow N_g'(t)$ is a Poisson Process with $\lambda_g' = 3\lambda_g = 10.8$
- $N_a'(t) :=$ # bonus from assists up to game $t \Rightarrow N_a'(t)$ is a Poisson Process with $\lambda_a' = 1\lambda_a = 2.4$
- $N'(t) := N_g'(t) + N_a'(t) \Rightarrow N'(t)$ is a Poisson Process with $\lambda' = \lambda_g' + \lambda_a' = 13.2$
- $N'(1) \sim \text{Poission}(\lambda') \Rightarrow \begin{cases} \mathbb{E}[N'(1)] = \lambda' = 13.2 \text{ K} \\ \text{SD}[N'(1)] = \sqrt{\lambda'} = \sqrt{13.2} \approx 3.63 \text{ K} \end{cases}$

Part (b)

- $\mathbb{P}\big(N_g(1) = 4, N_a(1) = 2\big) = \left[e^{-\lambda_g} \dfrac{\lambda_g^4}{4!}\right]\left[e^{-\lambda_a} \dfrac{\lambda_a^2}{2!}\right] = \left[e^{-3.6} \dfrac{3.6^4}{4!}\right]\left[e^{-2.4} \dfrac{2.4^2}{2!}\right] \approx 0.04996$

Part (c)

- $\mathbb{P}(\text{N}(0.5) = 4 | \text{N}(1) = 6) = \binom{6}{4}\left(\dfrac{0.5}{1}\right)^4 \left(1 - \dfrac{0.5}{1}\right)^{6-4} = 0.234375$

Question 7

5 Durrett 2.48 **10 / 10**

✓ **- 0 pts** **Correct**

- $\dfrac{\lambda_s}{\lambda_s + \lambda_m + \lambda_l} = \dfrac{0.24}{0.24 + 0.3 + 0.1} = \dfrac{3}{8}$

- Method 2
  - Define $T_s \sim \text{Exp}(\lambda_s), T_m \sim \text{Exp}(\lambda_m), T_l \sim \text{Exp}(\lambda_l)$
  - $\mathbb{P}(T_s < T_m, T_l) = \mathbb{P}(T_s < T_m < T_l) + \mathbb{P}(T_s < T_l < T_m)$

$$= \int_0^\infty f_{T_l}(l) \int_0^l f_{T_m}(m) \int_0^m f_{T_s}(s) ds\, dm\, dl + \int_0^\infty f_{T_m}(m) \int_0^m f_{T_l}(l) \int_0^l f_{T_s}(s) ds\, dl\, dm$$

$$= \frac{9}{32} + \frac{3}{32} = \frac{3}{8}$$

Question 6

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- $N'(1) \sim \text{Poission}(\lambda') \Rightarrow \begin{cases} \mathbb{E}[N'(1)] = \lambda' = 13.2 \text{ K} \\ \text{SD}[N'(1)] = \sqrt{\lambda'} = \sqrt{13.2} \approx 3.63 \text{ K} \end{cases}$

Part (b)

- $\mathbb{P}\left(N_g(1) = 4, N_a(1) = 2\right) = \left[e^{-\lambda_g} \frac{\lambda_g^4}{4!}\right]\left[e^{-\lambda_a} \frac{\lambda_a^2}{2!}\right] = \left[e^{-3.6} \frac{3.6^4}{4!}\right]\left[e^{-2.4} \frac{2.4^2}{2!}\right] \approx 0.04996$

Part (c)

- $\mathbb{P}(\text{N}(0.5) = 4 | \text{N}(1) = 6) = \binom{6}{4}\left(\frac{0.5}{1}\right)^4 \left(1 - \frac{0.5}{1}\right)^{6-4} = 0.234375$

Question 7

6 Durrett 2.49 **7 / 10**

✓ **- 3 pts** **(a) Conceptual errors. Examples: incorrect value for expectation or variance is used**

2.51. Two copy editors read a 300-page manuscript. The first found 100 typos, the second found 120, and their lists contain 80 errors in common. Suppose that the author's typos follow a Poisson process with some unknown rate λ per page, while the two copy editors catch errors with unknown probabilities of success p_1 and p_2. Let X_0 be the number of typos that neither found. Let X_1 and X_2 be the number of typos found only by 1 or only by 2, and let X_3 be the number of typos found by both. (a) Find the joint distribution of (X_0, X_1, X_2, X_3). (b) Use the answer to (a) to find an estimates of p_1, p_2 and then of the number of undiscovered typos.

Part (a)

- $N(t) :=$ # typos author made in the first t pages $\Rightarrow N(t)$ is a P.P with λ
- $N_0(t) :=$ # typos neither found in the first t pages $\Rightarrow N_0(t)$ is a P.P. with $\lambda_0 = \lambda(1-p_1)(1-p_2)$
- $N_1(t) :=$ # typos only 1 found in the first t pages $\Rightarrow N_1(t)$ is a P.P. with $\lambda_1 = \lambda p_1(1-p_2)$
- $N_2(t) :=$ # typos only 2 found in the first t pages $\Rightarrow N_2(t)$ is a P.P. with $\lambda_2 = \lambda p_2(1-p_1)$
- $N_3(t) :=$ # typos both found in the first t pages $\Rightarrow N_2(t)$ is a P.P. with $\lambda_3 = \lambda p_1 p_2$
- Since $N_0(t), N_1(t), N_2(t)$ and $N_3(t)$ are all thining of $N(t)$, they are mutually independet
- $X_0 = N_0(300) \sim \text{Poisson}\left(300\lambda(1-p_1)(1-p_2)\right)$
- $X_1 = N_1(300) \sim \text{Poisson}\left(300\lambda p_1(1-p_2)\right)$
- $X_2 = N_2(300) \sim \text{Poisson}\left(300\lambda p_2(1-p_1)\right)$
- $X_3 = N_3(300) \sim \text{Poisson}(300\lambda p_1 p_2)$
- $p(x_0, x_1, x_2, x_3) = p_{X_1}(x_1)p_{X_2}(x_2)p_{X_3}(x_3)p_{X_4}(x_4)$

$$= \left[e^{-300\lambda(1-p_1)(1-p_2)} \frac{\left(300\lambda(1-p_1)(1-p_2)\right)^{x_0}}{x_0!}\right] \cdot \left[e^{\lambda p_1(1-p_2)} \frac{\left(\lambda p_1(1-p_2)\right)^{x_1}}{x_1!}\right] \cdot$$

$$\left[e^{\lambda p_2(1-p_1)} \frac{\left(\lambda p_2(1-p_1)\right)^{x_2}}{x_2!}\right] \cdot \left[e^{\lambda p_1 p_2} \frac{(\lambda p_1 p_2)^{x_3}}{x_3!}\right] \text{ for } x_0, x_1, x_2, x_3 \in \mathbb{N}$$

Part (b)

- $\begin{cases} X_1 = 100 - 80 = 20 \\ X_2 = 120 - 80 = 40 \\ X_3 = 80 \end{cases} \Rightarrow \begin{cases} 300\lambda p_1(1-p_2) \approx 20 \\ 300\lambda p_2(1-p_1) \approx 40 \\ 300\lambda p_1 p_2 \approx 80 \end{cases} \Rightarrow \begin{cases} \hat{\lambda} = 1/2 \\ \hat{p}_1 = 2/3 \\ \hat{p}_2 = 4/5 \end{cases}$
- Estimated number of undiscovered typos $= 300\hat{\lambda}(1-\hat{p}_1)(1-\hat{p}_2) = 10$

7 Durrett 2.51 **10 / 10**

✓ **- 0 pts** Correct