Math 632 004 HW07

Shawn Zhong

TOTAL POINTS

30 / 30

QUESTION 1

1 Durrett 3.6 **10 / 10**

✓ **- 0 pts** **Correct**

- 2 pts (a) Computational mistake

- 4 pts (a) Conceptual error. Examples: does not set up a renewal as the sum of three turns

- 2 pts (b) Computational mistake

- 4 pts (b) Conceptual error. Examples: does not set up the process as a Markov chain correctly

QUESTION 2

2 Durrett 3.11 **10 / 10**

✓ **- 0 pts** **Correct**

- 2 pts (a) Computational mistake

- 4 pts (a) Conceptual error. Examples: computes expected reward without taking maximum

- 2 pts (b) Computational mistake

- 4 pts (b) Conceptual error. Examples: does not apply the memoryless property to get sleeping time distribution

QUESTION 3

3 Durrett 3.23 **10 / 10**

✓ **- 0 pts** **Correct**

- 3 pts Smaller errors. Examples: mistake in the integral or other computation

- 5 pts Substantial errors or several mistakes. Examples: uses wrong formula for $$g(z)$$ or $$E[Z(t)]$$ for large $$t$$

- 7 pts Serious errors. Examples: does not attempt to apply limiting theorems for renewals

- 10 pts No work submitted

HW7 - Problem

Tuesday, November 27, 2018 9:40 AM

Math 632 Lecture 4, Fall 2018, Homework 7

Due Tuesday December 4 by 10:05am

Question 1

Three children take turns shooting a ball at a basket. They each shoot until they miss and then it is next child's turn. Suppose that child i ($1 \le i \le 3$) makes a basket with probability p_i and that successive trials are independent. Use the following two approaches to determine the average fraction of time in the long run that child i shoots.

(a) (first method) Consider the renewal process with interarrival times $t_k = s_k^1 + s_k^2 + s_k^3$, where s_k^i is the number of shots child i takes in the k^{th} turn. Find $P(s_k^i = n)$. Then use this to compute the fraction of time child i spends shooting the ball.

(b) (second method) Consider the discrete time Markov Chain with three states, where the system is in state $i \in \mathcal{S} = \{1, 2, 3\}$ if child i has the ball. Use the theory from chapter 1 to determine the fraction of time child i spends shooting the ball in the long run.

Part (a)

- Let s_1^i, s_2^i ... be the time in state i, then this forms a alternating renewal process
- $\mathbb{P}(s_k^i = n) = p_i^{n-1}(1-p_i) \Rightarrow s_k^i \sim \text{Geo}(p_i) \Rightarrow \mathbb{E}[s_k^i] = \dfrac{1}{1-p_i}$
- By Alternating LLN, the fraction of time child i spends shooting the ball is $\dfrac{\frac{1}{1-p_i}}{\sum_{n=1}^{3}\frac{1}{1-p_n}}$

Part (b)

- Let π be a stationary measure for the Markov chain $\mathcal{P} = \begin{bmatrix} p_1 & 1-p_1 & 0 \\ 0 & p_2 & 1-p_2 \\ 1-p_3 & 0 & p_3 \end{bmatrix}$, then
- $\begin{cases} \pi(1) = \pi(1)p(1,1) + \pi(3)p(3,1) \\ \pi(2) = \pi(1)p(1,2) + \pi(2)p(2,2) \\ \pi(3) = \pi(2)p(2,3) + \pi(3)p(3,3) \\ \pi(1) + \pi(2) + \pi(3) = 1 \end{cases} \Rightarrow \begin{cases} \pi(1) = \pi(3)(1-p_3) + \pi(1)p_1 \\ \pi(2) = \pi(1)(1-p_1) + \pi(2)p_2 \\ \pi(3) = \pi(2)(1-p_2) + \pi(3)p_3 \\ \pi(1) + \pi(2) + \pi(3) = 1 \end{cases} \Rightarrow \pi(i) = \dfrac{\frac{1}{1-p_i}}{\sum_{n=1}^{3}\frac{1}{1-p_n}}$

Question 2

Durrett p. 115 Exercise 3.11(a)–(b) (doctor working at night). You can ignore (c).

Hint. The difference between parts (a) and (b) is a little subtle. The sleep cycle s_i in (b) is not the same as the reward r_i in the ith emergency cycle in (a). The first sleep cycle s_1 begins only when the doctor actually gets to sleep, in other words, when there is an emergency cycle t_i of length > 0.6.

1 Durrett 3.6 **10 / 10**

- ✓ **- 0 pts** **Correct**
- **- 2 pts** (a) Computational mistake
- **- 4 pts** (a) Conceptual error. Examples: does not set up a renewal as the sum of three turns
- **- 2 pts** (b) Computational mistake
- **- 4 pts** (b) Conceptual error. Examples: does not set up the process as a Markov chain correctly

3.11. A young doctor is working at night in an emergency room. Emergencies come in at times of a Poisson process with rate 0.5 per hour. The doctor can only get to sleep when it has been 36 minutes (.6 hours) since the last emergency. For example, if there is an emergency at 1:00 and a second one at 1:17 then she will not be able to get to sleep until at least 1:53, and it will be even later if there is another emergency before that time.
(a) Compute the long-run fraction of time she spends sleeping, by formulating a renewal reward process in which the reward in the ith interval is the amount of time she gets to sleep in that interval.
(b) The doctor alternates between sleeping for an amount of time s_i and being awake for an amount of time u_i. Use the result from (a) to compute Eu_i.

Part (a)

- Let t_i be the i-th interarrival time of emergency, then $t_i \sim \text{Exp}(0.5) \Rightarrow \mathbb{E}[t_i] = 2$
- Let $r_i = \max\{0, t_i - 0.6\}$ be the amount of time the doctor gets to sleep in the i-th interval
- $\mathbb{E}[r_i] = \int_0^\infty s \cdot f_{t_i}(s + 0.6) ds = \int_0^\infty s \cdot 0.5 e^{-0.5(s+0.6)} ds = \frac{2}{e^{0.3}}$
- Let $R(t) = \sum_{k=1}^{N(t)} r_i$ be the total amount of time the doctor gets to sleep up to time t
- By Renewal LLN, $\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[r_i]}{\mathbb{E}[t_i]} = \frac{1}{e^{0.3}}$

Part (b)

- $\mathbb{E}[s_i] = \mathbb{E}[t_i] = 2$ by memoryless property of exponential distribution
- By alternating LLN, $\frac{\mathbb{E}[s_i]}{\mathbb{E}[s_i] + \mathbb{E}[u_i]} = \lim_{t\to\infty} \frac{R(t)}{t} = \frac{1}{e^{0.3}}$
- Thus, $\mathbb{E}[u_i] = \mathbb{E}[s_i](e^{0.3} - 1) = 2(e^{0.3} - 1)$

Question 3

Durrett p. 117 Exercise 3.23 (taxi collects five people and then drives off). Letting Z denote the time until the onlooker sees a cab depart, find the density function g and the mean of Z. Compute the mean two ways: by (3.10) in Durrett, and by integrating with the density function g you found for Z, and this way confirm your answer. Assume that when the person comes to observe, the process has been running for a long time.

3.23. While visiting Haifa, Sid Resnick discovered that people who wish to travel from the port area up the mountain frequently take a shared taxi known as a sherut. The capacity of each car is 5 people. Potential customers arrive according to a Poisson process with rate λ. As soon as 5 people are in the car, it departs for The Carmel, and another taxi moves up to accept passengerso on. A local resident (who has no need of a ride) wanders onto the scene. What is the distribution of the time he has to wait to see a cab depart?

- Let $t = \tau_1 + \tau_2 + \tau_3 + \tau_4 + \tau_5 \sim \text{Gamma}(5, \lambda)$ be the interarrival time for a renewal process, then
  - $\mathbb{E}[t] = \frac{5}{\lambda}$

2 Durrett 3.11 **10 / 10**

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Part (a)

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- Let $r_i = \max\{0, t_i - 0.6\}$ be the amount of time the doctor gets to sleep in the i-th interval
- $\mathbb{E}[r_i] = \int_0^\infty s \cdot f_{t_i}(s + 0.6)ds = \int_0^\infty s \cdot 0.5e^{-0.5(s+0.6)}ds = \frac{2}{e^{0.3}}$
- Let $R(t) = \sum_{k=1}^{N(t)} r_i$ be the total amount of time the doctor gets to sleep up to time t
- By Renewal LLN, $\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[r_i]}{\mathbb{E}[t_i]} = \frac{1}{e^{0.3}}$

Part (b)

- $\mathbb{E}[s_i] = \mathbb{E}[t_i] = 2$ by memoryless property of exponential distribution
- By alternating LLN, $\frac{\mathbb{E}[s_i]}{\mathbb{E}[s_i] + \mathbb{E}[u_i]} = \lim_{t\to\infty} \frac{R(t)}{t} = \frac{1}{e^{0.3}}$
- Thus, $\mathbb{E}[u_i] = \mathbb{E}[s_i](e^{0.3} - 1) = 2(e^{0.3} - 1)$

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- Let $t = \tau_1 + \tau_2 + \tau_3 + \tau_4 + \tau_5 \sim \text{Gamma}(5, \lambda)$ be the interarrival time for a renewal process, then
    - $\mathbb{E}[t] = \frac{5}{\lambda}$

- $\mathbb{E}[t^2] = \text{Var}[t] + (E[t])^2 = \frac{5}{\lambda^2} + \left[\frac{5}{\lambda}\right]^2 = \frac{30}{\lambda^2}$
- $\mathbb{P}(t > z) = \int_z^\infty \lambda e^{-\lambda t} \frac{(\lambda t)^4}{4!} dt = e^{-\lambda z}\left[1 + \lambda z + \frac{(\lambda z)^2}{2} + \frac{(\lambda z)^3}{6} + \frac{(\lambda z)^4}{24}\right]$

- Thus, the limiting PDF of Z is
  - $g(z) = \frac{\mathbb{P}(t > z)}{\mathbb{E}[t]} = \frac{\lambda e^{-\lambda z}}{5}\left[1 + \lambda z + \frac{(\lambda z)^2}{2} + \frac{(\lambda z)^3}{6} + \frac{(\lambda z)^4}{24}\right]$
- The expectation of Z is
  - $\mathbb{E}[Z] = \int_0^\infty \frac{z\lambda e^{-\lambda z}}{5}\left[1 + \lambda z + \frac{(\lambda z)^2}{2} + \frac{(\lambda z)^3}{6} + \frac{(\lambda z)^4}{24}\right] dz = \frac{3}{\lambda}$
  - $\mathbb{E}[Z] = \frac{\mathbb{E}[t^2]}{2\mathbb{E}[t]} = \frac{30/\lambda^2}{10/\lambda} = \frac{3}{\lambda}$

3 Durrett 3.23 **10 / 10**

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