Math 632 004 HW08

Shawn Zhong

TOTAL POINTS

50 / 50

QUESTION 1

1 Problem 1 **10 / 10**

✓ **- 0 pts** **Correct**

- 1 pts (a) Smaller errors. Examples: forgot to separate the case for $$j=0$$

- 2 pts (a) Significant mistakes. Examples: did not include $$j$$ in the jump rates for $$q(j, j-1)$$

- 3 pts (b) Significant mistakes. Examples: very non-obvious steps are taken in the solution

- 2 pts (c) Incorrect

- 2 pts (d) Incorrect

QUESTION 2

2 Problem 2 **10 / 10**

✓ **- 0 pts** **Correct**

- 3 pts Smaller errors. Examples: computational mistake in the stationary distribution, forgot to address the $$\lambda=2\mu$$ case

- 5 pts Substantial errors or several mistakes. Examples: gets the rates wrong for the $$M/M/2$$ queue

- 7 pts Serious errors. Examples: uses the wrong formula entirely for finding a stationary distribution

- 10 pts No work submitted

QUESTION 3

3 Durrett 4.6 **10 / 10**

✓ **- 0 pts** **Correct**

- 2 pts (a) Computational mistake

- 4 pts (a) Conceptual error. Examples: does use the correct rates for the Markov chain, does not use a correct formula for the stationary distribution

- 2 pts (b) Computational mistake

- 4 pts (b) Conceptual error. Examples: does use the correct rates for the Markov chain, does not use a correct formula for the stationary distribution

QUESTION 4

4 Durrett 4.9 **10 / 10**

✓ **- 0 pts** **Correct**

- 3 pts Smaller errors. Examples: computational mistake in the stationary distribution

- 5 pts Substantial errors or several mistakes. Examples: gets the rates wrong for the Markov chain (wrong $$Q$$)

- 7 pts Serious errors. Examples: uses the wrong formula entirely for finding a stationary distribution

- 10 pts No work submitted

QUESTION 5

5 Durrett 4.38 **10 / 10**

✓ **- 0 pts** **Correct**

- 3 pts Smaller errors. Examples: computational mistake in the stationary distribution

- 5 pts Substantial errors or several mistakes. Examples: gets the rates wrong for the $$M/M/s$$ queue

- 7 pts Serious errors. Examples: uses the wrong formula entirely for finding a stationary distribution

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- 0 pts Correct

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- 10 pts No work submitted

HW8 - Problem

Saturday, December 8, 2018 10:35 PM

Math 632 Lecture 4, Fall 2018, Homework 8

Due Tuesday December 11 by 10:05am

Question 1

1. Consider an $M/M/\infty$ queue where customers arrive at rate λ, and the service time for each server is a rate μ exponential random variable. Let $X(t)$ denote the number of customers in the queue at time t. Assume that $X(0) = 0$.

(a) State the Kolmogorov forward equations for this process.

(b) Set $M(t) = E[X(t)]$. Prove that

$$\frac{dM}{dt} = \lambda - \mu M(t).$$

Hint. You will need the solution to (a), but do not attempt to solve them. Feel free to differentiate any series you see term by term.

(c) Solve the differential equation for $M(t)$.

(d) Evaluate $\lim_{t\to\infty} M(t)$. The stationary distribution for $X(t)$ is given in Example 4.16 of Durrett's book. Compare the limit you found to the expected value of the stationary distribution.

Part (a)

- The jump rates are $\begin{cases} q(n, n+1) = \lambda & \forall n \geq 0 \\ q(n, n-1) = n\mu & \forall n \geq 1 \end{cases}$
- The rate out of state n is $\lambda_n = \lambda + n\mu, \forall n \geq 0$
- The forward Kolmogorov equations are
  - $p_t'(i,j) = \begin{cases} p_t(i,j-1)\lambda + p_t(i,j+1)(j+1)\mu - p_t(i,j)(\lambda + j\mu) & j \geq 1 \\ p_t(i,1)\mu - p_t(i,0)\lambda & j = 0 \end{cases}$

Part (b)

- $M(t) = \mathbb{E}[X(t)] = \sum_{j=1}^{\infty} j \cdot \mathbb{P}(X(t) = j) = \sum_{j=1}^{\infty} j \cdot p_t(0,j)$
- $M'(t) = \sum_{j=1}^{\infty} j \cdot p_t'(0,j)$

$$= \sum_{j=1}^{\infty} j \cdot \left(p_t(0,j-1)\lambda + p_t(0,j+1)(j+1)\mu - p_t(0,j)(\lambda + j\mu)\right)$$

$$= \sum_{j=1}^{\infty} j\lambda \cdot p_t(0,j-1) + \sum_{j=1}^{\infty} (j^2\mu + j\mu) \cdot p_t(0,j+1) - \sum_{j=1}^{\infty} (j\lambda + j^2\mu) \cdot p_t(0,j)$$

$$= \sum_{j=0}^{\infty}(j+1)\lambda \cdot p_t(0,j) + \sum_{j=2}^{\infty}\left((j-1)^2\mu + (j-1)\mu\right) \cdot p_t(0,j) - \sum_{j=1}^{\infty}(j\lambda + j^2\mu) \cdot p_t(0,j)$$

$$= \lambda p_t(0,0) + 2\lambda p_t(0,1) - (\lambda+\mu)p_t(0,j)$$

$$+ \sum_{j=2}^{\infty}\left[(j+1)\lambda + (j-1)^2\mu + (j-1)\mu - (j\lambda + j^2\mu)\right] \cdot p_t(0,j)$$

$$= \lambda p_t(0,0) + \lambda p_t(0,1) - \mu p_t(0,j) + \sum_{j=2}^{\infty}(\lambda - j\mu) \cdot p_t(0,j)$$

$$= \lambda p_t(0,0) + \sum_{j=1}^{\infty}(\lambda - j\mu) \cdot p_t(0,j)$$

$$= \lambda \sum_{j=0}^{\infty} p_t(0,j) - \mu \sum_{j=1}^{\infty} j \cdot p_t(0,j)$$

$$= \lambda - \mu M(t)$$

Part (c)

- $M'(t) + \mu M(t) = \lambda$
- $\underbrace{e^{\mu t}M'(t) + e^{\mu t}\mu M(t)}_{(e^{\mu t}M(t))'} = \lambda e^{\mu t}$
- $e^{\mu t}M(t) = \int \lambda e^{\mu t} dt = \frac{\lambda}{\mu}e^{\mu t} + C$
- Thus, $M(t) = \frac{\lambda}{\mu} + Ce^{-\mu t}$ for some constant $C \in \mathbb{R}$
- With initial value $M(0) = \mathbb{E}[X(0)] = 0$, we have $C = -\frac{\lambda}{\mu}$
- Therefore, $M(t) = \frac{\lambda - e^{-\mu t}}{\mu}$

Part (d)

- $\lim_{t\to\infty} M(t) = \frac{\lambda}{\mu}$, which is consistent with Example 4.16

Question 2

2. Consider an $M/M/2$ queue where customers arrive at rate λ and the rate for each server is μ. However, arriving customers who see N customers already in the line ahead of them leave and never return, for some $N \geq 2$. Let $X(t)$ denote the number of customers in the system at time t. Find the stationary distribution for $X(t)$.

- The jump rates are $\begin{cases} q(n, n+1) = \lambda & \forall n \in \{0, \dots, N-1\} \\ q(n, n-1) = 2\mu & \forall n \in \{2, \dots, N\} \\ q(1,0) = \mu & \end{cases}$
- Let π be a stationary distribution for $X(t)$ that safisfy the detailed balanced equations
- Then $\pi(i) = \frac{\prod_{k=0}^{i-1} q(k, k+1)}{\prod_{k=1}^{i} q(k, k-1)}\pi(0) = \frac{\lambda^i}{2^{i-1}\mu^i}\pi(0)$ for $1 \leq i \leq N$

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$$= \sum_{j=0}^{\infty}(j+1)\lambda \cdot p_t(0,j) + \sum_{j=2}^{\infty}\left((j-1)^2\mu + (j-1)\mu\right) \cdot p_t(0,j) - \sum_{j=1}^{\infty}(j\lambda + j^2\mu) \cdot p_t(0,j)$$

$$= \lambda p_t(0,0) + 2\lambda p_t(0,1) - (\lambda+\mu)p_t(0,j)$$

$$+ \sum_{j=2}^{\infty}\left[(j+1)\lambda + (j-1)^2\mu + (j-1)\mu - (j\lambda + j^2\mu)\right] \cdot p_t(0,j)$$

$$= \lambda p_t(0,0) + \lambda p_t(0,1) - \mu p_t(0,j) + \sum_{j=2}^{\infty}(\lambda - j\mu) \cdot p_t(0,j)$$

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$$= \lambda \sum_{j=0}^{\infty} p_t(0,j) - \mu \sum_{j=1}^{\infty} j \cdot p_t(0,j)$$

$$= \lambda - \mu M(t)$$

Part (c)

- $M'(t) + \mu M(t) = \lambda$
- $\underbrace{e^{\mu t}M'(t) + e^{\mu t}\mu M(t)}_{(e^{\mu t}M(t))'} = \lambda e^{\mu t}$
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- Let π be a stationary distribution for $X(t)$ that safisfy the detailed balanced equations
- Then $\pi(i) = \frac{\prod_{k=0}^{i-1} q(k,k+1)}{\prod_{k=1}^{i} q(k,k-1)}\pi(0) = \frac{\lambda^i}{2^{i-1}\mu^i}\pi(0)$ for $1 \leq i \leq N$

- $1 = \sum_{i=0}^{N} \pi(i) = \left[1 + \sum_{i=1}^{N} \frac{\lambda^i}{2^{i-1}\mu^i}\right]\pi(0) = \frac{\lambda + 2\mu - 2^{1-N}\lambda^{1+N}\mu^{-N}}{2\mu - \lambda}\pi(0)$
- $\Rightarrow \pi(0) = \frac{2\mu - \lambda}{\lambda + 2\mu - 2^{1-N}\lambda^{1+N}\mu^{-N}}$
- Thus, $\pi(i) = \begin{cases} \frac{2\mu - \lambda}{\lambda + 2\mu - 2^{1-N}\lambda^{1+N}\mu^{-N}} & i = 0 \\ \frac{\lambda^i(2\mu - \lambda)}{2^{i-1}\mu^i(\lambda + 2\mu - 2^{1-N}\lambda^{1+N}\mu^{-N})} & 1 \le i \le N \end{cases}$

Question 3

4.5. Two people are working in a small office selling shares in a mutual fund. Each is either on the phone or not. Suppose that salesman i is on the phone for an exponential amount of time with rate μ_i and then off the phone for an exponential amount of time with rate λ_i. (a) Formulate a Markov chain model for this system with state space $\{0, 1, 2, 12\}$ where the state indicates who is on the phone. (b) Find the stationary distribution.

4.6. (a) Consider the special case of the previous problem in which $\lambda_1 = \lambda_2 = 1$, and $\mu_1 = \mu_2 = 3$, and find the stationary probabilities. (b) Suppose they upgrade their telephone system so that a call to one line that is busy is forwarded to the other phone and lost if that phone is busy. Find the new stationary probabilities.

Part (a)

- The jump rates are $\begin{cases} q(0,1) = q(1,12) = q(0,2) = q(2,12) = 1 \\ q(1,0) = q(12,1) = q(2,0) = q(12,2) = 3 \end{cases}$
- The out rates are $\lambda_0 = 2, \lambda_1 = \lambda_2 = 4, \lambda_{12} = 6$
- Thus the jump rate matrix is $Q = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 3 & -4 & 0 & 1 \\ 3 & 0 & -4 & 1 \\ 0 & 3 & 3 & -6 \end{bmatrix}$
- Let π be a stationary distribution, then
- $\begin{cases} \pi Q = 0 \\ \sum_{i \in S} \pi(i) = 1 \end{cases} \Rightarrow \begin{cases} -2\pi(0) + 3\pi(1) + 3\pi(2) = 0 \\ \pi(0) - 4\pi(1) + 3\pi(12) = 0 \\ \pi(0) - 4\pi(2) + 3\pi(12) = 0 \\ \pi(1) + \pi(2) - 6\pi(12) = 0 \\ \pi(0) + \pi(1) + \pi(2) + \pi(12) = 1 \end{cases} \Rightarrow \pi = \begin{bmatrix} \frac{9}{16} & \frac{3}{16} & \frac{3}{16} & \frac{1}{16} \end{bmatrix}$

Part (b)

- The jump rate matrix is $Q = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 3 & -5 & 0 & 2 \\ 3 & 0 & -5 & 2 \\ 0 & 3 & 3 & -6 \end{bmatrix}$
- Let π be a stationary distribution, then

2 Problem 2 **10 / 10**

✓ **- 0 pts** Correct

- 3 pts Smaller errors. Examples: computational mistake in the stationary distribution, forgot to address the $$\lambda=2\mu$$ case

- 5 pts Substantial errors or several mistakes. Examples: gets the rates wrong for the $$M/M/2$$ queue

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- $1 = \sum_{i=0}^{N} \pi(i) = \left[1 + \sum_{i=1}^{N} \frac{\lambda^i}{2^{i-1}\mu^i}\right] \pi(0) = \frac{\lambda + 2\mu - 2^{1-N}\lambda^{1+N}\mu^{-N}}{2\mu - \lambda} \pi(0)$
- $\Rightarrow \pi(0) = \frac{2\mu - \lambda}{\lambda + 2\mu - 2^{1-N}\lambda^{1+N}\mu^{-N}}$
- Thus, $\pi(i) = \begin{cases} \frac{2\mu - \lambda}{\lambda + 2\mu - 2^{1-N}\lambda^{1+N}\mu^{-N}} & i = 0 \\ \frac{\lambda^i(2\mu - \lambda)}{2^{i-1}\mu^i(\lambda + 2\mu - 2^{1-N}\lambda^{1+N}\mu^{-N})} & 1 \le i \le N \end{cases}$

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- The out rates are $\lambda_0 = 2, \lambda_1 = \lambda_2 = 4, \lambda_{12} = 6$
- Thus the jump rate matrix is $Q = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 3 & -4 & 0 & 1 \\ 3 & 0 & -4 & 1 \\ 0 & 3 & 3 & -6 \end{bmatrix}$
- Let π be a stationary distribution, then
- $\begin{cases} \pi Q = 0 \\ \sum_{i \in S} \pi(i) = 1 \end{cases} \Rightarrow \begin{cases} -2\pi(0) + 3\pi(1) + 3\pi(2) = 0 \\ \pi(0) - 4\pi(1) + 3\pi(12) = 0 \\ \pi(0) - 4\pi(2) + 3\pi(12) = 0 \\ \pi(1) + \pi(2) - 6\pi(12) = 0 \\ \pi(0) + \pi(1) + \pi(2) + \pi(12) = 1 \end{cases} \Rightarrow \pi = \begin{bmatrix} \frac{9}{16} & \frac{3}{16} & \frac{3}{16} & \frac{1}{16} \end{bmatrix}$

Part (b)

- The jump rate matrix is $Q = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 3 & -5 & 0 & 2 \\ 3 & 0 & -5 & 2 \\ 0 & 3 & 3 & -6 \end{bmatrix}$
- Let π be a stationary distribution, then

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Question 4

4.9. A hemoglobin molecule can carry one oxygen or one carbon monoxide molecule. Suppose that the two types of gases arrive at rates 1 and 2 and attach for an exponential amount of time with rates 3 and 4, respectively. Formulate a Markov chain model with state space $\{+, 0, -\}$ where $+$ denotes an attached oxygen molecule, $-$ an attached carbon monoxide molecule, and 0 a free hemoglobin molecule and find the long-run fraction of time the hemoglobin molecule is in each of its three states.

- The jump rate matrix is $Q = \begin{bmatrix} -3 & 3 & 0 \\ 1 & -3 & 2 \\ 0 & 4 & -4 \end{bmatrix}$
- Let π be a stationary distribution, then
- $\begin{cases} \pi Q = 0 \\ \sum_{i \in S} \pi(i) = 1 \end{cases} \Rightarrow \begin{cases} -3\pi(+) + \pi(0) = 0 \\ 3\pi(+) - 3\pi(0) + 4\pi(-) = 0 \\ 2\pi(0) - 4\pi(0) = 0 \\ \pi(+) + \pi(0) + \pi(-) = 1 \end{cases} \Rightarrow \pi = \begin{bmatrix} \frac{2}{11} & \frac{6}{11} & \frac{3}{11} \end{bmatrix}$

Question 5

4.38. Consider an $M/M/s$ queue with no waiting room. In words, requests for a phone line occur at a rate λ. If one of the s lines is free, the customer takes it and talks for an exponential amount of time with rate μ. If no lines are free, the customer goes away never to come back. Find the stationary distribution. You do not have to evaluate the normalizing constant.

- The jump rates are $\begin{cases} q(n, n+1) = \lambda \\ q(n, n-1) = n\mu \end{cases}$
- Let π be a stationary distribution that safisfy the detailed balanced equations
- $\pi(i) = \frac{\prod_{k=0}^{i-1} q(k, k+1)}{\prod_{k=1}^{i} q(k, k-1)} \pi(0) = \frac{\lambda^i}{i!\, \mu^i} \pi(0)$

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- 2 pts (a) Computational mistake

- 4 pts (a) Conceptual error. Examples: does use the correct rates for the Markov chain, does not use a correct formula for the stationary distribution

- 2 pts (b) Computational mistake

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- $\begin{cases} \pi Q = 0 \\ \sum_{i\in S} \pi(i) = 1 \end{cases} \Rightarrow \begin{cases} -2\pi(0) + 3\pi(1) + 3\pi(2) = 0 \\ \pi(0) - 5\pi(1) + 3\pi(12) = 0 \\ \pi(0) - 5\pi(2) + 3\pi(12) = 0 \\ 2\pi(1) + 2\pi(2) - 6\pi(12) = 0 \\ \pi(0) + \pi(1) + \pi(2) + \pi(12) = 1 \end{cases} \Rightarrow \pi = \begin{bmatrix} \frac{9}{17} & \frac{3}{17} & \frac{3}{17} & \frac{2}{17} \end{bmatrix}$

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