10/2

Monday, October 2, 2017

Question 1

- Is $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} \pi \\ \sqrt{2} \\ -1 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix} \right\}$ independent?
- Claim
    - If S is linearly dependent
- Proof
    - If S is linearly independent, then
        - $\dim(\text{span}(S)) = |S| = 5$
    - But because $\text{span}(S)$ is a subspace of $\mathbb{R}^4$
        - $\dim(\text{span}(S)) \leq \dim \mathbb{R}^4 = 4$
    - So S is linearly dependent

Question 2

- Prove
    - $1, \sin x, \sin 2x$ is linearly independent
- Claim
    - $\forall\, a, b, c \in \mathbb{R}$
    - if $a + b \cdot sin\, x + sin\, 2x = 0, \qquad \forall x \in [0,1]$
    - then $a = b = c = 0$
- Proof
    - Set $x = 0 \;\Rightarrow a = 0$
    - Set $x = \frac{\pi}{6} \;\Rightarrow \frac{1}{2} b + \frac{\sqrt{3}}{2} c = 0$
    - Set $x = \frac{\pi}{4} \;\Rightarrow b = c = 0$
    - Therefore $a = b = c = 0$

10/4

Wednesday, October 4, 2017

Proof Writing

- Question
  - Let V be a vector space
  - Let $x, y \in V$ such that $\{x, y\}$ is independent
  - Prove that $\{2x + y, 3x + 2y\}$ is independent
- Proof
  - Let $c_1, c_2 \in \mathbb{R}$ be arbitrary constant
  - $c_1(2x + y) + c_2(3x + 2y) = 0$
  - $(2c_1 + 3c_2)x + (c_1 + 2c_2)y = 0$
  - Let $\begin{cases} d_1 = 2c_1 + 3c_2 \\ d_2 = c_1 + 2c_2 \end{cases}, d_1, d_2 \in \mathbb{R}$
  - $d_1 x + d_2 y = 0$
  - Because $\{x, y\}$ is independent
  - $d_1 = d_2 = 0$
  - $\begin{cases} d_1 = 2c_1 + 3c_2 = 0 \\ d_2 = c_1 + 2c_2 = 0 \end{cases} \Rightarrow c_1 = c_2 = 0$
  - Therefore $\{2x + y, 3x + 2y\}$ is independent
- Prompt
  - Exchange proofs with someone else. In a different color of pen or pencil, give them written feedback on their proof.
  - The main things to be looking for are:
    - Is the proof logically valid?
    - Is the proof understandable and clearly written?
    - Is the proof well-organized?
  - Here are some more questions it might be useful to ask (but don't feel like you're limited to these or have to answer all of them):
    - Is it clear from the start what's being proved?
    - Was there any point where you were confused or had to fill in some gaps?
    - Are all the statements precise, or are there vague or ambiguous phrases?
    - Is there a clear distinction made between assumptions, claims, statements that are a consequence of something shown earlier, and theorems being cited?
    - Has every variable been defined before it's used?

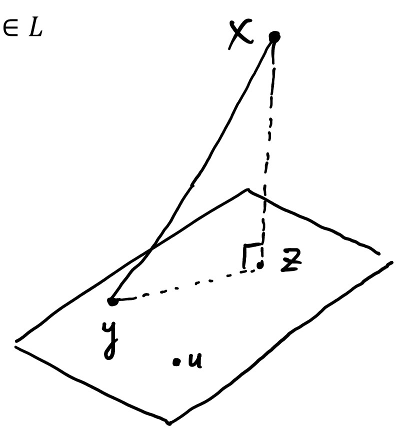
- Is everything quantified that needs quantifiers? (Are the quantifiers in the right order?)
- Is the proof written in grammatically correct, complete sentences?
- If any definitions are stated, are they correctly stated?
- Are all the steps in the right order?
- Is the proof convincing?

10/5

Thursday, October 5, 2017

Best Approximation of Elements

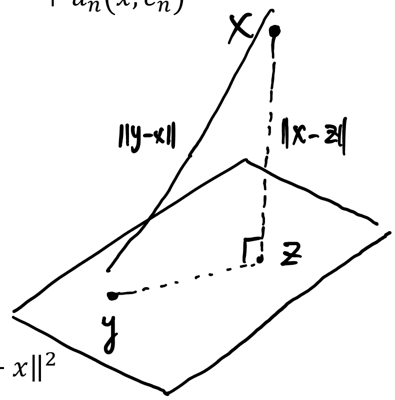
- Theorem
  - V: vector space with inner product
  - $L \subseteq V$: finite dimensional linear subspace
  - If $x \in V$ then there exists excatly one $z \in L$
  - that minimizes the distance to x
  - i.e. $\forall y \in L,\ \|y - x\| \geq \|z - x\|$ and
  - If $y \neq z$ then $\|y - x\| > \|z - x\|$
- Solution
  - L is finite dimensional therefore it has a basis
  - Gram-Schmidt says that we can assume the basis is orthonormal
  - i. e. L has a basis $\{e_1, e_2, \dots, e_n\}$ where $\begin{cases} (e_k, e_l) = 0 & k \neq l \\ (e_k, e_k) = 1 & \forall k \end{cases}$
  - Then z is given by $z = (x, e_1)e_1 + (x, e_2)e_2 + \cdots + (x, e_n)e_n$
  - Since z is a linear combination of $\{e_1, \dots, e_n\}$, $z \in L$
- Claim
  - $x - z$ is perpendicular to all $u \in L$
  - i.e. if $u \in L$ then $u \perp x - z$
  - i. e. $(u, x - z) = 0$
  - i. e. $(u, x) = (u, z)$



- Proof: $(u, x) = (u, z)$
  - Let $u \in L$ be given
  - Then $\{e_1, \dots e_n\}$ is a basis for L
  - So for certain $u_1, \dots, u_n \in \mathbb{R}$
  - Calculate (u, x)
    - $(u, x) = (u_1 e_1 + \cdots + u_n e_n, x)$
    - $= u_1(e_1, x) + \cdots + u_n(e_n, x)$
  - Calculate (u, z)
    - $(u, z) = (u_1 e_1 + \cdots + u_n e_n, (x, e_1)e_1 + \cdots + (x, e_n)e_n)$
    - $= [u_1(x_1, e_1)(e_1, e_1) + \cdots + u_1(x_1, e_n)(e_1, e_n)] + \cdots$
      $+ [u_n(x_1, e_1)(e_n, e_1) + \cdots + u_n(x_n, e_n)(e_n, e_n)]$
    - $= u_1(x, e_1) + u_n(x, e_2) + \cdots + u_n(x, e_n)$ ✓✓

$+[u_n(x_1, e_1)(e_n, e_1) + \cdots + u_n(x_n, e_n)(e_n, e_n)]$

- $= u_1(x, e_1) + u_n(x, e_2) + \cdots + u_n(x, e_n)$



- Therefore $(u, x) = (u, z)$
- i.e. $u \perp x - z, \ \forall u \in L$

- Proof: $\forall y \in L, \|y - x\| \geq \|z - x\|$
  - Let $y \in L$ be given
  - $\begin{cases} y - x = (y - z) + (z - x) \\ y - z \perp z - x \end{cases}$
  - $\Rightarrow \|y - x\|^2 = \|y - z\|^2 + \|z - x\|^2$
  - $\Rightarrow \|y - x\|^2 \geq \|z - x\|^2$
  - $\Rightarrow \|y - x\| \geq \|z - x\|$
  - Also if $y \neq z$ then $\|y - x\| > \|z - x\|$

Foorier Series

- $V = \{\text{all continuous function } f: [0, \pi] \to \mathbb{R}\}$
- $(f, g) = \int_0^\pi f(x) g(x) dx$
- Let $f_n(x) = \sin(nx)$
- $\Rightarrow (f_n, f_m) = \int_0^\pi \sin(nx) \sin(mx) \, dx$

10/9

Monday, October 9, 2017

Question

- Let V be a finite-dimensional inner product space
- $S \subseteq V$ is a subspace of V
- Let $S^{\perp} = \{v \in V | \forall s \in S, \langle v, s \rangle = 0\}$
- Prove $(S^{\perp})^{\perp} = S$

Answer: First, $(\mathbf{S}^{\perp})^{\perp}$ is the orthogonal complement of $\mathbf{S}^{\perp}$, which is itself the orthogonal complement of $\mathbf{S}$, so $(\mathbf{S}^{\perp})^{\perp} = \mathbf{S}$ means that $\mathbf{S}$ is the orthogonal complement of its orthogonal complement.

To show that it is true, we want to show that $\mathbf{S}$ is contained in $(\mathbf{S}^{\perp})^{\perp}$ and, conversely, that $(\mathbf{S}^{\perp})^{\perp}$ is contained in $\mathbf{S}$; if we can show both containments, then the only possible conclusion is that $(\mathbf{S}^{\perp})^{\perp} = \mathbf{S}$.

To show the first containment, suppose $\mathbf{v} \in \mathbf{S}$ and $\mathbf{w} \in \mathbf{S}^{\perp}$. Then

$$\langle \mathbf{v}, \mathbf{w} \rangle = 0$$

by the definition of $\mathbf{S}^{\perp}$. Thus, $\mathbf{S}$ is certainly contained in $(\mathbf{S}^{\perp})^{\perp}$ (which consists of all vectors in $\mathbb{R}^n$ which are orthogonal to $\mathbf{S}^{\perp}$).

To show the other containment, suppose $\mathbf{v} \in (\mathbf{S}^{\perp})^{\perp}$ (meaning that $\mathbf{v}$ is orthogonal to all vectors in $\mathbf{S}^{\perp}$); then we want to show that $\mathbf{v} \in \mathbf{S}$. I'm sure there must be a better way to see this, but here's one that works. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be a basis for $\mathbf{S}$ and let $\{\mathbf{w}_1, \ldots, \mathbf{w}_q\}$ be a basis for $\mathbf{S}^{\perp}$. If $\mathbf{v} \notin \mathbf{S}$, then $\{\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}\}$ is a linearly independent set. Since each vector in that set is orthogonal to all of $\mathbf{S}^{\perp}$, the set

$$\{\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}, \mathbf{w}_1, \ldots, \mathbf{w}_q\}$$

is linearly independent. Since there are $p+q+1$ vectors in this set, this means that $p+q+1 \leq n$ or, equivalently, $p+q \leq n-1$. On the other hand, if A is the matrix whose ith row is u_i^T, then the row space of A is $\mathbf{S}$ and the nullspace of A is $\mathbf{S}^{\perp}$. Since $\mathbf{S}$ is p-dimensional, the rank of A is p, meaning that the dimension of $\text{nul}(A) = \mathbf{S}^{\perp}$ is $q = n - p$. Therefore,

$$p + q = p + (n - p) = n,$$

contradicting the fact that $p + q \leq n - 1$. From this contradiction, then, we see that, if $\mathbf{v} \in (\mathbf{S}^{\perp})^{\perp}$, it must be the case that $\mathbf{v} \in \mathbf{S}$.

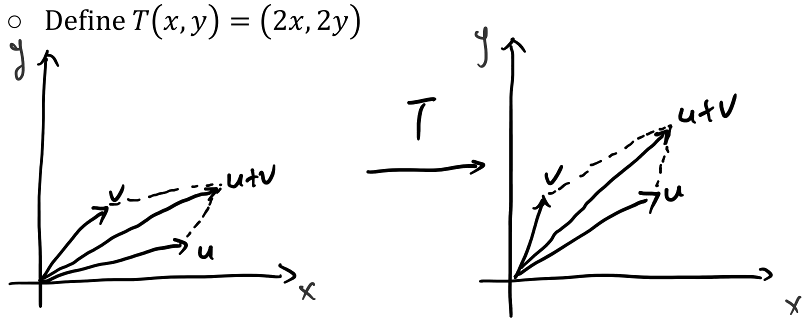
10/10

Tuesday, October 10, 2017

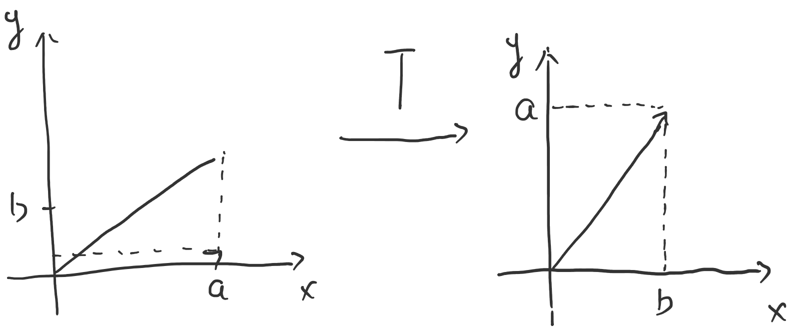
Linear Transformations

- Definition
  - Let V and W be two vector spaces
  - Then a map/function/transformation/mapping
  - $T: V \to W$ is called linear if
  - $\begin{cases} T(x+y) = T(x) + T(y) & \forall x, y \in V \\ T(c \cdot x) = c \cdot T(x) & \forall x \in V, c \in \mathbb{R} \end{cases}$
- Mapping notation
  - In the mapping $T: V \to W$
  - V is called "domain"
  - W is called "codomain" or "target set"
  - $T(v)$ must be defined $\forall v \in V$
  - $T(v)$ always belongs to W
- Example 1
  - Let V, W be any vector space
  - Define $Tx = 0, \ \forall x \in V$
  - $\begin{cases} T(x+y) = 0 \\ T(x) + T(y) = 0 + 0 = 0 \end{cases} \Rightarrow T(x+y) = Tx + Ty$
  - $\begin{cases} T(c \cdot x) = 0 \\ c \cdot T(x) = c \cdot 0 = 0 \end{cases} \Rightarrow T(c \cdot x) = c \cdot T(x)$
  - Therefore this mapping is a linear transformation
- Example 2
  - Let V, W be any vector space
  - Define $Tv = w \neq 0, \ \forall v \in V$
  - $T(x) + T(y) = 2w \neq w = T(x+y)$
  - Therefore this mapping is not a linear transformation
- Example 3
  - Let $V = W$ be the same vector space
  - Define $Tx = x, \forall v \in V$
  - Then T is a linear transformation
  - T is called the identity map from V to V
  - Common notations: $\mathrm{id}, \mathrm{id}_{\mathrm{V}}, 1_{\mathrm{V}}$
- Example 4

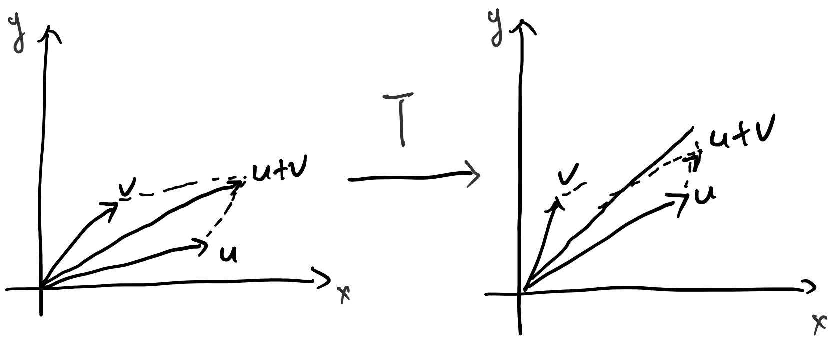
    - Let $V = W = \mathbb{R}^2$ be the same vector space
    - Define $T(x, y) = (2x, 2y)$



    - $T(u) + T(v) = 2u + 2v = 2(u + v) = T(u + v)$
    - $T(c \cdot u) = 2c \cdot u = c \cdot (2u) = c \cdot T(u)$
    - Therefore T is a linear transformation
- Example 5
    - Let $V = W = \mathbb{R}^2$ be the same vector space
    - Define $T(a, b) = (b, a)$



    - It's reflection in the diagonal
- Example 6
    - Let $V = W = \mathbb{R}^2$ be the same vector space
    - Define $Tu = u$ rotated by 30° counter-clockwise



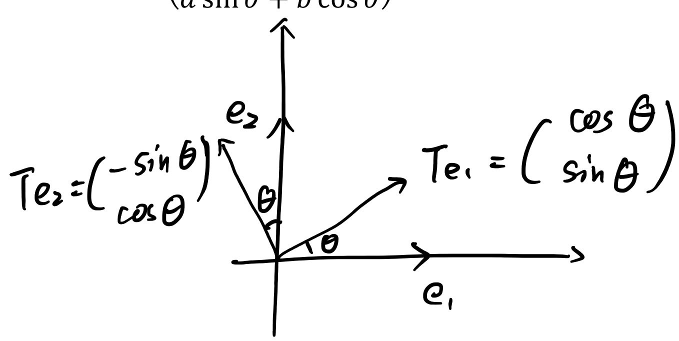
    - Proof by graph $T(u + v) = T(u) + T(v)$
    - We can also prove that $T(c \cdot v) = c \cdot T(v)$
    - Therefore T is a linear transformation

Linear Transformation on Basis

- Theorem

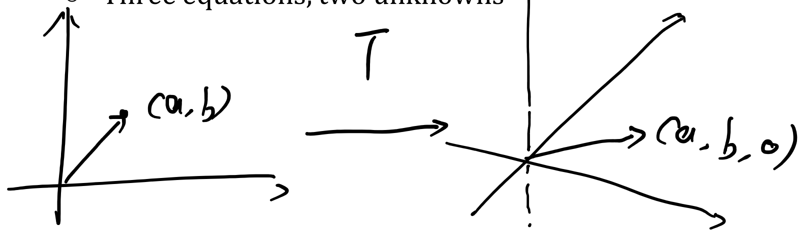
- Suppose $T: V \to W$ is a linear transformation
- Let $\{e_1, \dots, e_n\}$ be a basis for V
- Then T is completely defermined by
- $\{Te_1,, Te_2 \dots, Te_n\}$
- Suppose we known $Te_1, Te_2, \dots, Te_n$,
- and let $x \in V$ be given
- Then there are $c_1, c_2, \dots, c_n \in \mathbb{R}$
- such that $x = c_1e_1 + c_2e_2 + \cdots + c_ne_n$, then
- $T(x) = T(c_1e_1 + c_2e_2 + \cdots + c_ne_n)$
- $= T(c_1e_1) + T(c_2e_2) + \cdots + T(c_ne_n)$
- $= c_1Te_1 + c_2Te_2 + \cdots c_nTe_n$

- Example (Rotation)
  - Let $V = W = \mathbb{R}^2$ be the same vector space
  - Define T rotate by θ counter-clockwise
  - Pick a basis $\{e_1, e_2\}$, where
    - $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
    - $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
  - Compute Te_1, Te_2
    - $Te_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$
    - $Te_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$
  - Compute $T(ae_1 + be_2)$
    - $T(ae_1 + be_2)$
    - $= aTe_1 + bTe_2$
    - $= a\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + b\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$
    - $= \begin{pmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \end{pmatrix}$



Solving System of Equations

- Setup
  - $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_n \end{cases}$
  - Define a transformation $T\colon\ \mathbb{R}^n \to \mathbb{R}^n$
  - Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$
  - Then $Tx = y$ is a linear transformation
- Property of one-to-one map
  - A linear map $T: V \to W$ is a one-to-one map
  - if for all $u, v \in V$
  - $Tu = Tv \Rightarrow u = v$
  - i.e. The equation $Tx = y$ has at most one solution
- Example of one-to-one map
  - Let $V = \mathbb{R}^2, W = \mathbb{R}^3$
  - $T(x_1, x_2) = (x_1, x_2, 0) = (y_1, y_2, y_3)$
  - $\begin{cases} 1x_1 + 0x_2 = y_1 \\ 0x_1 + 1x_2 = y_2 \\ 0x_1 + 0x_2 = y_3 \end{cases} \Rightarrow \begin{cases} y_1 = x_1 \\ y_2 = x_2 \\ y_3 = 0 \end{cases}$
  - Three equations, two unknowns



- Theorem
  - A linear map $T: V \to W$ is injective
  - if for all $x \in V$
  - $Tx = 0 \Rightarrow x = 0$

10/11

Wednesday, October 11, 2017

Question

- Given
  - $V = C([-1,1])$
  - $\langle v, w \rangle = \int_{-1}^{1} v(x)w(x)dx$
- Find the linear polynomial closest to $f(x) = e^x$
- Answer
  - Let $S = \text{span}\{1, x\}$
  - Projection of f onto S is
  - $\frac{\langle 1, e^x \rangle}{\langle 1,1 \rangle} \cdot 1 + \frac{\langle x, e^x \rangle}{\langle x, x \rangle} \cdot x$
  - Therefore the linear polynomial closest to $f(x) = e^x$ is
  - $g(x) = \frac{3}{e}x + \frac{e - e^{-1}}{2}$

10/12

Thursday, October 12, 2017

Injective

- Definition
  - If V, W are vector space and $T: V \to W$ is linear
  - Then T is injective if for all $x, y \in V$
  - $Tx = Ty \Rightarrow x = y$
- Theorem
  - $T: V \to W$ is injective if and only if for all $x \in V$
  - $Tx = 0 \Rightarrow x = 0$
  - i.e. if and only if $N(T) = \{0\}$
- Proof
  - Suppose $Tx = 0 \Rightarrow x = 0$ for all $x \in V$
  - Let $x, y \in V$ be giben, and assume
    - $Tx = Ty$
  - Since T is linear, we have
    - $T(x - y) = Tx - Ty = 0$
  - Therefore
    - $x - y = 0$
    - $\Rightarrow x = y$

Null Space

- Definition
  - If $T: V \to W$ is linear then
  - $Null(T) = N(T) = kern(T) \stackrel{\text{def}}{=} \{x \in V | Tx = 0\}$
- Theorem
  - $N(T)$ is a linear subspace of V
- Proof:
  - To prove $N(T) \in V$ is a linear subspace
  - We need to check closure properties i.e.
    - $x, y \in N(T) \Rightarrow x + y \in N(T)$
    - $x \in N(T), c \in \mathbb{R} \Rightarrow cx \in N(T)$
  - Check closure under addition
    - Let $x, y \in N(T)$, then $Tx = 0, Ty = 0$
    - We have $T(x + y) = Tx + Ty = 0 + 0 = 0$

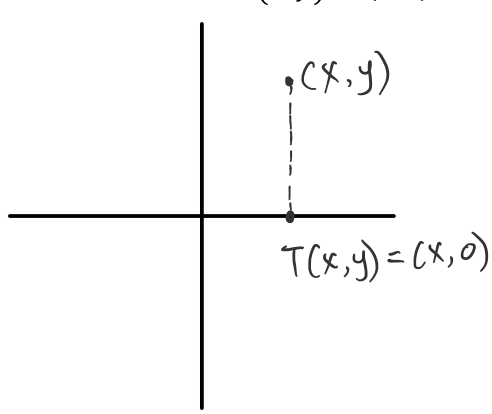
        - Therefore $x + y \in N(T)$
    - Check closure under scalar multiplication
        - Let $x \in N(T)$, then $Tx = 0$
        - Let $c \in \mathbb{R}$, then $T(cx) = c \cdot Tx = c \cdot 0 = 0$
        - Therefore $cx \in N(T)$
    - In conclusion, $N(T) \in V$ is a linear subspace

Range

- Definition
    - If $T: V \to W$ is linear then
    - $Range(T) = R(T) = \{Tx | x \in V\}$
- Theorem
    - $R(T)$ is a linear subspace of W

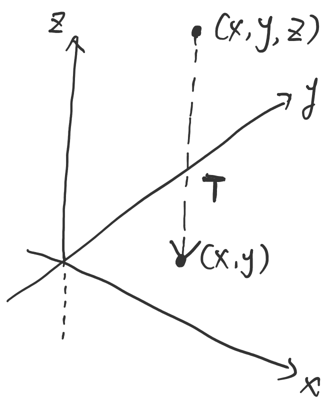
Examples

- Example 1
    - Let $V = W = \mathbb{R}^2,\ T(x, y) = (x, y)$
    - Injective?
        - Given $(x, y) \in \mathbb{R}^2$, and $(\bar{x}, \bar{y}) \in \mathbb{R}^2$
        - with $T(x, y) = T(\bar{x}, \bar{y})$
        - By definition of T
        - $(x, y) = (\bar{x}, \bar{y})$
        - So T is injective
    - Null Space?
        - Because T is injective
        - $N(T) = \{0,0\}$
    - Range?
        - $R(T) \overset{\text{def}}{=} \{T(x, y) | (x, y) \in \mathbb{R}^2\} = \mathbb{R}^2$
- Example 2
    - Let $V = W = \mathbb{R}^2,\ T(x, y) = (x, 0)$

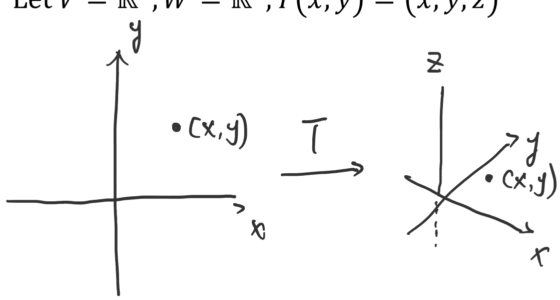


- Injective?
  - No
  - $T(1,0) = T(1,1) = (1,0)$
- Null Space?
  - $N(T) = \{u|Tu = 0\} = \{(0,t)|t \in \mathbb{R}\}$
- Range?
  - $R(T) = \{T(x,y)|(x,y) \in \mathbb{R}^2\}$
  - $= \{(t,0)|t \in \mathbb{R}^2\}$
  - $= x$-axis

* Example 3
  - Let $V = \mathbb{R}^3, W = \mathbb{R}^2, T(x,y,z) = (x,y)$



  - Injective?
    - No
    - $T(1,1,0) = T(1,1,1) = (1,1)$
  - Null Space?
    - $N(T) = \{(0,0,t)|t \in \mathbb{R}\}$
  - Range?
    - $R(T) = \{T(x,y,z)|(x,y,z) \in \mathbb{R}^3\}$
    - $= \{(x,y)|(x,y) \in \mathbb{R}^2\} = \mathbb{R}^2$
* Example 4
  - Let $V = \mathbb{R}^2, W = \mathbb{R}^3, T(x,y) = (x,y,z)$



  - T is injective
  - $N(T) = \{0,0\}$

- $R(T) = \{(x, y, 0) | (x, y) \in \mathbb{R}^2\} = xy$-plane

- Summary

T	V	W	$N(T)$	$\dim N(T)$	$R(T)$	$\dim R(T)$
$T(x,y) = (x,y)$	$\mathbb{R}^2$	$\mathbb{R}^2$	$\{0\}$	0	$\mathbb{R}^2$	2
$T(x,y) = (x,0)$	$\mathbb{R}^2$	$\mathbb{R}^2$	y-axis	1	x-axis	1
$T(x,y,z) = (x,y)$	$\mathbb{R}^3$	$\mathbb{R}^2$	z-axis	1	$\mathbb{R}^2$	2
$T(x,y) = (x,y,z)$	$\mathbb{R}^2$	$\mathbb{R}^3$	$\{0\}$	0	xy-plane	2

Rank–Nullity Theorem

- Statement
  - If $T: V \to W$ is linear and if V is finite dimensional
  - Then $\dim \mathrm{N}(T) + \dim \mathrm{R}(T) = \dim V$
- Proof
  - Let
    - $\dim \mathrm{N}(T) = k$
    - $\dim V = n$
    - $\{e_1, \dots, e_k\}$ be a basis for $N(T)$
  - Claim
    - $\{e_1, \dots, e_k\} \subseteq V$ is independent
    - $\Rightarrow$ There is a basis $\{e_1, \dots, e_k, e_{k+1}, \dots, e_n\}$ of V so $\dim V = n$
    - $\{Te_{k+1}, Te_{k+2}, \dots, Te_n\}$ is a basis for $R(T)$
  - Prove $\{Te_{k+1}, Te_{k+2}, \dots, Te_n\}$ is independent
    - Suppose
      - $c_{k+1}Te_{k+1} + \cdots + c_nTe_n = 0$
    - Then
      - $T(c_{k+1}e_{k+1} + \cdots + c_ne_n) = 0$
      - $\Rightarrow c_{k+1}e_{k+1} + \cdots + c_ne_n \in N(T)$
    - Since $\{e_1, \dots, e_k\}$ is a basis for $N(T)$
      - $c_{k+1}e_{k+1} + \cdots + c_ne_n = c_1e_1 + \cdots + c_ke_k$
      - $-c_1e_1 - \cdots - c_ke_k + c_{k+1}e_{k+1} + \cdots + c_ne_n = 0$
    - Since $\{e_1, \dots, e_n\}$ is independent
      - $c_1 = c_2 = \cdots = c_n = 0$
    - In particular
      - $c_{k+1}Te_{k+1} + \cdots + c_nTe_n = 0$
      - implies $c_{k+1} = c_{k+2} = \cdots = c_n = 0$
    - Therefore
      - $\{Te_{k+1}, Te_{k+2}, \dots, Te_n\}$ is independent

- Prove $\{Te_{k+1}, Te_{k+2}, \dots, Te_n\}$ spans $R(T)$
  - Every $y \in R(T)$ is of the form
    - $y = Tx$
    - For some $x \in V$
  - $\{e_1, \dots, e_n\}$ is a basis for V, so
    - $x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$
    - For some $x_1, x_2, \dots, x_n \in \mathbb{R}$
  - Therefore
    - $y = Tx$
    - $= T(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n)$
    - $= x_1 Te_1 + \cdots + x_k Te_k + x_{k+1} Te_{k+1} + \cdots + x_n Te_n$
    - $= x_{k+1} Te_{k+1} + \cdots + x_n Te_n \in \text{span}\{Te_{k+1}, Te_{k+2}, \dots, Te_n\}$
- Conclusion
  - $\dim R(T) = n - k = \dim V - \dim N(T)$
  - $\Rightarrow \dim \mathrm{N}(T) + \dim \mathrm{R}(T) = \dim V$

10/16

Monday, October 16, 2017

Question 1

- Given
  - Let V be a set
  - Let $S, T: V \to V$ be invertible functions
- Prove
  - ST is also invertible and $(ST)^{-1} = T^{-1}S^{-1}$
- Proof
  - $(ST)(T^{-1}S^{-1}) = S(TT^{-1})S^{-1} = SIS^{-1} = SS^{-1} = I$
  - $(T^{-1}S^{-1})(ST) = T^{-1}(S^{-1}S)T = T^{-1}IT = T^{-1}T = I$

Question 2

- Given
  - Let V and W be finite-dimensional vector spaces.
- Proof
  - There exists a surjective linear map $f: V \to W$ if and only if $\dim W \leq \dim V$

10/17

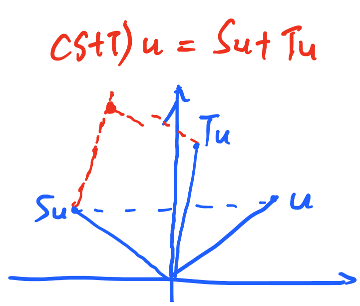
Tuesday, October 17, 2017

Examples of Linear Transformations

- Example 1
  - $V = \{\text{all polynomials}\}$
  - Consider $D: V \to V$ defined by
    - Given $f \in V$
    - $Df = g$ if $g(x) = f'(x)$
    - e.g. $D(1 + x - 3x^2) = 1 - 6x$
  - Null Space
    - $\text{Null}(D) = \{f \in V | Df = 0\}$
    - $= \{f \in V | f'(x) = 0\}$
    - $= \{f \in V | f \text{ is constant function}\}$
    - $= \{f(x) = c | c \in R\}$
    - $\dim \text{Null}(D) = 1$
    - Basis for $\text{Null}(D) = \{1\}$
- Example 2
  - $V = \{\text{all polynomials}\}$
  - $K: V \to V$
  - $Kf = g \Leftrightarrow g(x) = \int_0^x f(s)ds$
  - e.g. $K(x^2 + 3) = \int_0^x f(s^2 + 3)ds = \left[\frac{s^2}{3} + 3s\right]_0^x = \frac{1}{3}x^3 + 3x$

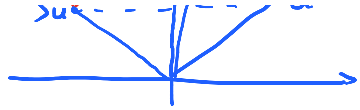
Addition and Scalar Multiplication of Linear Transformations

- Addition
  - V, W: vector spaces
  - $T, S: V \to W$: linear transformations
  - $T + S$ is the map $V \to W$ with $(T + S)(x) = Tx + Sx$
- Example
  - $V = W = \mathbb{R}^2$
  - $T =$ rotation by 45° counter-clockwise
  - $S =$ reflection in the y-axis
  - $T + S = ?$



- Theroem

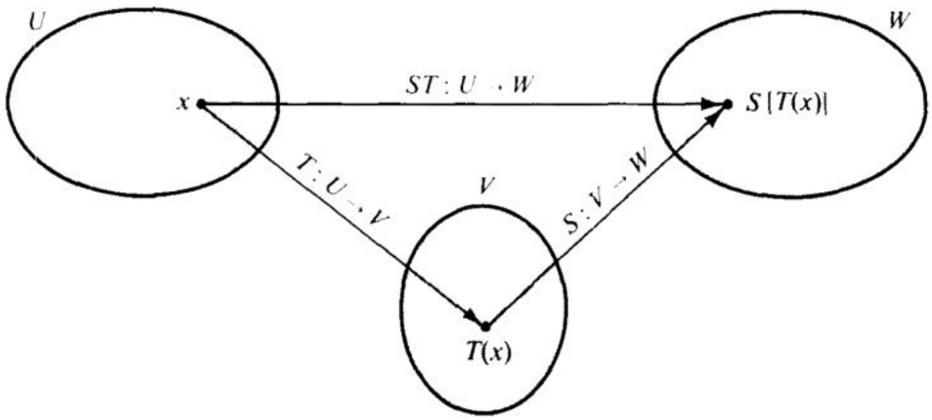
- $T + S = ?$



- Theroem
  - Statement
    - If $T, S: V \to W$ is linear, so are $(T + S)$
  - Proof: closed under addition
    - $(T + S)(x + y)$
    - $= T(x + y) + S(x + y)$
    - $= Tx + Ty + Sx + Sy$
    - $= (Tx + Sx) + (Ty + Sy)$
    - $= (T + S)(x) + (T + S)(y)$
  - Proof: closed under scalar multiplication
    - $(T + S)(cx)$
    - $= T(cx) + S(cx)$
    - $= c \cdot T(x) + c \cdot S(x)$
    - $= c[T(x) + S(x)]$
    - $= c(T + S)(x)$
- Scalar Multiplication
  - V, W: vector spaces
  - $T, S: V \to W$: linear transformations
  - $cT: V \to W$ $(c \in \mathbb{R})$ is defined by
  - $(cT)(x) = c(Tx),\ \forall x \in V$
- Theorem
  - Let V, W be two vector spaces
  - $\mathcal{L}(V, W) = \{$all linear transformation from V to $W\}$
  - Then $\mathcal{L}(V, W)$ is a vector space
  - e.g. $T, S \in \mathcal{L}(V, W) \Rightarrow c_1 T + c_2 S \in \mathcal{L}(V, W),\ \forall c_1, c_2 \in \mathbb{R}$

Multiplication/Composition of Linear Transformations

- Definition
  - U, V, W: vector spaces
  - $T: U \to V, \qquad S: V \to W$
  - Then $ST: V \to W$ is given by $(ST)(x) = S(Tx)$



- Theorem
  - If S, T_1, T_2 is linear, then $S(T_1 + T_2) = ST_1 + ST_2$
- Example
  - Given
    - $V = \{\text{all polynomials}\}$
    - $D, K: V \to V$
    - $Df = f', (Kf)(x) = \int_0^x f(s)ds$
  - $DKf = ?$
    - Let $g = Kf = \int_0^x f(s)ds$
    - $D(g(x)) = \frac{d}{dx} g(x) = \frac{d}{dx} \int_0^x f(s)ds = f(x)$
    - Therefore $DKf = f$
  - $KDf = ?$
    - $KDf = \int_0^x (Df)(s)ds = \int_0^x f'(s)ds = f(x) - f(0)$
    - Therefore $KDf \neq f$

Injective and Inverse

- Injective
  - T is injective if and only if $N(T) = \{0\}$
  - If $T: V \to W$ is injective then
  - $Tx = y$ has exactly one solution for every $y \in \text{Range}(T)$
  - ($\text{Range}(T) = \{Tx | x \in V\}$, "exactly one" because T is injective)
- Inverse
  - $T^{-1}: \text{Range}(T) \to V$ is given by
  - $T^{-1}(y) = x, \qquad \text{if } y = Tx$
- Example
  - Given

        - $V = \mathbb{R}^2, \qquad W = \mathbb{R}^2$
        - $T: V \to W$
        - $Tx = (x, x)$
    - Whether T is inversable?
        - $Tx = 0 \Rightarrow x = 0 \Rightarrow N(T) = \{0\}$
        - $\text{Range}(T) = \{(x, x) | x \in \mathbb{R}\} = \{(x, y) \in \mathbb{R}^2 | x = y\}$
        - $T^{-1}: \text{Range}(T) \to \mathbb{R}$
        - $T^{-1}(x, x) = x$
- Theorem
    - Statement
        - $T^{-1}: \text{Range}(T) \to V$ is linear $\Leftrightarrow \begin{cases} T^{-1}(u + v) = T^{-1}(u) + T^{-1}(v) \\ T^{-1}(c \cdot u) = c \cdot T^{-1}(u) \\ \forall u, v \in \text{Range}(T), c \in \mathbb{R} \end{cases}$
    - Proof
        - If $u \in \text{Range}(T)$ then there is an $x \in V$ with $u = Tx$
        - By definition of T^{-1}, $x = T^{-1}(u)$
        - Similarly, there is $y \in V$ with $v = Ty$, and $y = T^{-1}(u)$
        - $T(x + y) = Tx + Ty = u + v$
        - $\Rightarrow u + v \in \text{Range}(T)$
        - $\Rightarrow x + y = T^{-1}(u + v)$
- Theorem
    - Statement
        - Suppose V is a finite-dimensional linear space
        - $T: V \to V$ is injective, then
        - $\text{Range}(T) = V$
    - Proof
        - Rank–Nullity Theorem says that
        - $\dim \text{Null}(T) + \text{rank}(T) = \dim V$
        - T is injective $\Rightarrow \text{Null}(T) = \{0\} \Rightarrow \dim \text{Null}(T) = 0$
        - Therefore $\dim \text{Range}(T) = \dim V$
        - Also, $\text{Range}(T)$ is a sunspace of V
        - $\Rightarrow \text{Range}(T) = V$
- Theorem
    - Suppose V is a finite-dimensional linear space
    - $T: V \to V$ is injective, then
    - $Tx = y$ has a unique solution for every $y \in V$
- Example

- Given
  - $V = \{\text{all polynomials}\}$
  - $D, K: V \to V$
  - $Df = f', (Kf)(x) = \int_0^x f(s)ds$
- Is K injective?
  - We have proven $DKf = f$
  - Suppose $Kf = 0$, then $D(Kf) = 0$
  - But $f = DKf$, so $f = 0$
  - $\Rightarrow K$ is injective
- Is K surjective?
  - Suppose K is surjective then
  - Given $g \in V$, we can solve $Kf = g$ with $f \in W$
  - i.e. given $g \in V$, there is one f with
  - $\int_0^x f(s)ds = 1, \qquad \forall x \in \mathbb{R}$
  - At $x = 0$, we have
  - $\int_0^0 f(s)ds = 1$
  - Which makes a contradiction, therefore K is not surjective

10/18

Wednesday, October 18, 2017

Theorem

- V, W: vector spaces
- $x_1, \dots, x_n$: basis for V
- For any $w_1, \dots, w_n \in W$
- There is a unique linear map $T: V \to W$
- s.t. $\begin{cases} T(x_1) = w_1 \\ \vdots \\ T(x_n) = w_n \end{cases}$
- $v \in W \Rightarrow \exists c_1, \dots, c_n \in \mathbb{R}$
- $\text{s.t.}\, v = c_1 x_1 + \cdots + c_n x_n$
- $T(v) = c_1 w_1 + \cdots + c_n w_n$
- Linear map can be determined only by operations on basis

Question 1

- Requirement
    - $T\colon \mathbb{R}^2 \to \mathbb{R}^3$
    - $\dim(\text{range}(T)) = 1$
- Example
    - $T(x, y) = (x, 0,0)$
    - $T(x, y) = (0, y, y)$
    - $T(x, y) = (0, x + 3y, -2x - 6y)$

Question 2

- Requirement
    - $T\colon \mathbb{R}^2 \to \mathbb{R}^2$
    - $S\colon \mathbb{R}^2 \to \mathbb{R}^2$
    - $ST = -TS$
- Example
    - $T(x, y) = (-y, x)$
    - $S(x, y) = (-x, y)$

Question 3

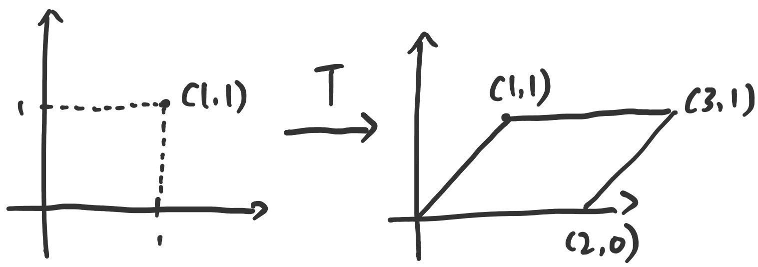
- Requirement
    - $T\colon \mathbb{R}^3 \to \mathbb{R}^3$
    - $T^2 \neq 0$

- $T^3 \neq 0$

- Example
  - $T(x, y, z) = (0, x, y)$

Question 4

- Requirement
  - $T\colon \mathbb{R}^2 \to \mathbb{R}^2$
  - T maps the unit square to the parallgram below



  - $T(0,0) = (0,0)$
  - $T(1,0) = (2,0)$
  - $T(0,1) = (1,1)$
  - $T(1,1) = (3,0)$

- Example
  - $T(x, y) = (2x + y, y)$
  - $T(x, y) = (2y + x, x)$

Question 5

- Requirement
  - $T\colon \mathbb{R}^3 \to \mathbb{R}^3$
  - $T(x, 0,0) = (2x, 0,0)$
  - $T^3(0, a, b) = (0, a, b)$

- Example
  - $T(x, y, z) = (2x, y, z)$

Question 6

- Requirement
  - $T\colon \mathbb{R}^2 \to \mathbb{R}^2$
  - $T(1,0) = (1,0)$
  - $\{(x, y), T(x, y)\}$ is independent whenever $y \neq 0$

- Example
  - $T(x, y) = (x + y, y)$

Question 7

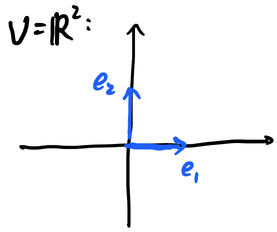
- Requirement
    - $T: \mathbb{R}^2 \to \mathbb{R}^3$
    - T is injective
    - $\dim(\text{range}(T)) = 1$
- Example
    - Impossible

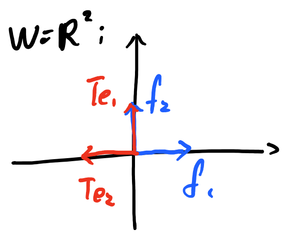
10/19

Thursday, October 19, 2017

Matrix Representation of Linear Transformations

- Given
  - Linear Transformation $T: V \to W$
  - Basis for V: $\{e_1, \dots, e_n\}$
  - Basis for W: $\{f_1, \dots, f_m\}$
- Let $x \in V, y = T(x)$ then
  - $\begin{cases} x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n \\ y = y_1 f_1 + y_2 f_2 + \cdots + y_m f_m \end{cases}$
- $T(e_k) \in W \Rightarrow T(e_k)$ is a linear combination of $\{f_1, \dots, f_m\}$ i. e.
  - $T(e_k) = \sum_{i=1}^{m} T_{ik} f_i = T_{1k} f_1 + T_{2k} f_2 + \cdots + T_{mk} f_m$
- Suppose we know T_{ik} $(i \in \{1, \dots m\}, k \in \{1, \dots, n\})$, then
  - $T(x) = T(x_1 e_1 + \cdots + x_n e_n)$
  - $= x_1(T_{11} f_1 + \cdots + T_{m1} f_m) + \cdots + x_n(T_{1n} f_1 + \cdots + T_{mn} f_m)$
  - $= (T_{11} x_1 + \cdots + T_{1n} x_n) f_1 + \cdots + (T_{m1} x_1 + \cdots + T_{mn} x_n) f_m$
  - $= y_1 f_1 + \cdots + y_m f_m$
  - where $y_i = T_{i1} x_1 + \cdots + T_{1n} x_n$
  - Note: $T(e_k) = T_{1k} f_1 + T_{2k} f_2 + \cdots + T_{mk} f_m$
- The matrix of the linear transformation $T: V \to W$ is
  - $\text{Mat}(T, \{e\}, \{f\}) = \begin{bmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{m1} & \cdots & T_{mn} \end{bmatrix}$
  - with respect to the basis $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_m\}$ of V and W
- Example
  - $V = W = \mathbb{R}^2$
  - e, f: standard basis for V and W
  - T: rotation by 90°
  - $Te_1 = 0 \cdot f_1 + 1 \cdot f_2$
  - $Te_2 = (-1) \cdot f_1 + 0 \cdot f_2$
  - $\text{mat}(T) = [Te_1, Te_2] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$





Matrix Multiplication

- Motivation

- Consider the composition of linear transformations T and S
- $U \xrightarrow{T} V \xrightarrow{S} W$
- basis for U: $\{e_1, \dots, e_k\}$
- basis for V: $\{f_1, \dots, e_l\}$
- basis for W: $\{g_1, \dots, e_m\}$
- $\mathrm{mat}(ST) = \mathrm{mat}(S) \cdot \mathrm{mat}(T)$

- Definition
  - $A_{m\times n}B_{n\times q} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nq} \end{bmatrix}$
  - $= \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{n1} & \cdots & a_{11}b_{1q} + \cdots + a_{1n}b_{nq} \\ \vdots & \ddots & \vdots \\ a_{m1}b_{11} + \cdots + a_{mn}b_{n1} & \cdots & a_{m1}b_{1q} + \cdots + a_{mn}b_{nq} \end{bmatrix}_{m\times q}$
- Example
  - T: rotation by 90°
  - $\mathrm{mat}(T) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
  - $\begin{cases} T^2e_1 = -f_1 = (-1)\cdot f_1 + 0 \cdot f_2 \\ T^2e_2 = -f_2 = 0 \cdot f_1 + (-1)\cdot f_2 \end{cases} \Rightarrow \mathrm{mat}(T^2) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
  - $(\mathrm{mat}(T))^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
  - Therefore $\mathrm{mat}(T^2) = (\mathrm{mat}(T))^2$

10/23

Monday, October 23, 2017

Question 1

- Given
  - Let V and W be finite-dimensional vector spaces.
- Proof
  - There exists a surjective linear map $f: V \to W$ if and only if $\dim W \leq \dim V$
- Prove: $\exists$surjective linear map $f: V \to W \Rightarrow \dim W \leq \dim V$
  - $\dim V = \dim N(f) + \dim R(f)$
  - f is surjective $\Rightarrow \dim R(f) = \dim W$
  - $\dim V = \dim N(f) + \dim W$
  - $\dim V \geq \dim W$
- Prove: $\dim W \leq \dim V \Rightarrow \exists$surjective linear map $f: V \to W$
  - $\{e_1, \dots, e_n\}$: basis for V
  - $\{g_1, \dots, g_m\}$: basis for W
  - Construct linear map f where
    - $f(e_1) = g_1$
    - $f(e_2) = g_2$
    - $\vdots$
    - $f(e_m) = g_m$
    - $f(e_{m+1}) = 0$
    - $f(e_{m+2}) = 0$
    - $\vdots$
    - $f(e_n) = 0$
  - Obviously, f is surjective

Question 2

- Given
  - Define a linear map $T: \mathbb{R}^3 \to \mathbb{R}^2$ as follows
  - $T(i) = (0,0), T(j) = (1,1), T(k) = (1,-1)$
  - where i, j, k is the standard basis of $\mathbb{R}^3$
- Question (a)
  - Compute $T(4i - j + k)$ and determine the nullity and rank of T
  - $T(4i - j + k) = 4T(i) - T(j) + T(k) = 4(0,0) - (1,1) + (1,-1) = (0,-2)$

- $R(T) = \{c_1 T(i) + c_2 T(j) + c_3 T(k) | c_1, c_2, c_3 \in \mathbb{R}\} = \mathbb{R}^2$
  - rank $= \dim R(T) = 2$
  - nullity $= \dim \mathbb{R}^3 -$ rank $= 1$
- Question (b)
  - Determine the matrix of T
  - $m(T) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$
- Question (c)
  - Determine the matrix of T using the same basis on the domain
  - and the basis $(1,1), (1,2)$ on the codomain
  - $m(T) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix}$

10/24

Tuesday, 24 October 2017

Matrix Representation of Linear Transformation

- Definition
  - $T: V \to W$
  - $\{e_1, e_2 \dots e_n\}$: basis for V
  - $\{f_1, f_2 \dots f_n\}$: basis for W
  - $matrix(T, \{e_k\}, \{f_l\}) = m(T) = \begin{bmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{m1} & \cdots & T_{mn} \end{bmatrix}$
- Example
  - $\begin{cases} Te_1 = T_{11}f_1 + \cdots + T_{m1}f_m \\ Te_2 = T_{12}f_1 + \cdots + T_{m2}f_m \\ \vdots \\ Te_n = T_{1n}f_1 + \cdots + T_{mn}f_m \end{cases}$ a

Algebra of Linear Transformations vs. Algebra of Matrices

- Comparison

Linear Transformations	Matrices
$T + S$	$m(T + s) = m(T) + m(S)$
$c \cdot T$	$m(cT) = c \cdot m(T)$
$S \circ T$	$m(S \circ T) = m(S) \cdot m(T)$

- Proof: $m(S \circ T) = m(S)m(T)$
  - Setup
    - $T: U \to V, \qquad S: V \to W$
    - $\{e_1 \dots e_n\}$: basis of U
    - $\{f_1 \dots f_m\}$: basis of V
    - $\{g_1 \dots g_k\}$: basis of W
  - Let $m(R) = m(S \circ T), \qquad$ where $R = S \circ T$
  - $m(T)$ is defined by
    - $\begin{cases} Te_1 = T_{11}f_1 + \cdots + T_{m1}f_m \\ Te_2 = T_{12}f_1 + \cdots + T_{m2}f_m \\ \vdots \\ Te_n = T_{1n}f_1 + \cdots + T_{mn}f_m \end{cases}$
  - $m(S)$ is defined by
    - $\begin{cases} Sf_1 = S_{11}g_1 + \cdots + S_{k1}g_k \\ Sf_2 = S_{12}g_1 + \cdots + S_{k2}g_k \\ \vdots \\ Sf_m = S_{1m}g_m + \cdots + S_{km}g_k \end{cases}$

- $m(R)$ is defined by
  - $\begin{cases} Re_1 = R_{11}e_1 + \cdots + R_{k1}g_k \\ Re_2 = R_{12}e_1 + \cdots + R_{k2}g_k \\ \vdots \\ Re_n = R_{1n}e_1 + \cdots + R_{kn}g_k \end{cases}$
- R_{ij} = Coefficient of g_i in Re_j = Coefficient of g_i in $(S \circ T)e_j$
- Expanding $(S \circ T)e_j$, we have
  - $(S \circ T)e_j = S\left(Te_j\right)$
  - $= S\left(T_{1j}f_1 + T_{2j}f_2 + \cdots + T_{mj}f_m\right)$
  - $= T_{1j} \cdot Sf_1 + T_{2j} \cdot Sf_2 + \cdots + T_{mj} \cdot Sf_m$
  - $= T_{1j}(S_{11}g_1 + \cdots + S_{k1}g_k) + T_{2j}(S_{12}g_1 + \cdots + S_{k2}g_k) + \cdots + T_{mj}(S_{1m}g_1 + \cdots + S_{km}g_k)$
- Terms containing g_i
  - $T_{1j}S_{i1}g_i + T_{2j}S_{i2}g_i + \cdots + T_{mj}S_{im}g_i$
  - $= \left(S_{i1}T_{1j} + S_{i2}T_{2j} + \cdots + S_{im}T_{mj}\right)g_i$
- Therefore
  - $R = \left(R_{ij}\right)_{i,j=1}^{n,k} = \left(S_{i1}T_{1j} + S_{i2}T_{2j} + \cdots + S_{im}T_{mj}\right)_{i,j=1}^{n,k}$
  - $m(S)m(T) = \begin{bmatrix} S_{11} & \cdots & S_{1m} \\ \vdots & \ddots & \vdots \\ S_{k1} & \cdots & S_{km} \end{bmatrix} \times \begin{bmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{m1} & \cdots & T_{mn} \end{bmatrix} = \left(S_{i1}T_{1j} + S_{i2}T_{2j} + \cdots + S_{im}T_{mj}\right)_{i,j=1}^{n,k}$
  - $\Rightarrow m(S \circ T) = m(S)m(T)$

Matrix Multiplication

- Example
  - $V = W = \mathbb{R}^2$ with standard basis
  - T = rotation by θ
    - $m(T) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
  - S = rotation by φ
    - $m(S) = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$
  - S = rotation by $\theta + \varphi$
    - $m(ST) = \begin{bmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) \end{bmatrix}$
    - $m(S)m(T) = \begin{bmatrix} \cos\varphi\ \cos\theta - \sin\varphi\sin\theta & -\sin\theta\cos\varphi - \sin\varphi\cos\theta \\ \sin\theta\cos\varphi + \sin\varphi\cos\theta & \cos\varphi\ \cos\theta - \sin\varphi\sin\theta \end{bmatrix}$
    - Therefore $m(ST) = m(S)m(T)$
- Example: $T \neq 0$, but $T^2 = 0$

- $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad T(x,y) = (0,x)$
- $\Rightarrow T^2 = T \times T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \times 1 + 1 \times 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
- Note: $T \neq 0$, but $T^2 = 0$

- Example: $ST \neq TS$
  - $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$
  - $TS = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
  - $ST = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
  - Note: $ST \neq TS$
  - Therefore matrix multiplication is not commutative
- Example
  - $S, T: V \to V, \qquad$ (or S, T are square matrice)
  - $(S+T)^2 = (S+T)(S+T) = S^2 + ST + TS + T^2$
  - Note: $(S+T)^2 \neq S^2 + 2TS + T^2 \neq S^2 + 2ST + T^2$

Solving Linear Equations using Matrix

- Matrix representation of Linear Equations
  - $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = y_m \end{cases} \Leftrightarrow \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$
  - Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$
  - Then the linear equations could by represented as $Ax = y$
- Row reduction
  - Multiply an equation with $c \neq 0$
  - Switch equations
  - Subtract one equation from anther
- Example
  - Question
    - $\begin{cases} x_1 + x_2 + x_3 = 5 \\ 2x_1 - x_2 + x_3 = 7 \end{cases}$
  - Convert into Matrix
    - $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & -1 & 1 & 7 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 1/3 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2/3 & 4 \\ 0 & 1 & 1/3 & 1 \end{array}\right]$
  - Substitute back

- $$\begin{cases} x_1 = 4 - \frac{2}{3}x_3 \\ x_2 = 1 - \frac{1}{3}x_3 \\ x_3 \in \mathbb{R} \end{cases}$$

- Let $x_3 = 3t$, then the general solution is
  - $$\begin{bmatrix} 4 - 2t \\ 1 - t \\ 3t \end{bmatrix}, t \in \mathbb{R}$$

10/25

Wednesday, October 25, 2017

Question 1

- $T: \mathbb{R}^3 \to \mathbb{R}^2$ with T defined as
  - $T(i) = (0,0)$
  - $T(j) = (1,1)$
  - $T(k) = (1,-1)$
- Find the matrix for normal basis
  - $M(T, \{i, j, k\}, \{i, j\}) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$
- Find the matrix using $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ as the basis for $\mathbb{R}^2$
  - $M\left(T, \{i, j, k\}, \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix}$
- Find bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ so that the matrix is diagonal
  - $M\left(T, \left\{\begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}, \{i, j\}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
  - $M(T, \{j, k, i\}, \{T(i), T(k)\}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Question 2

- Let T: $\mathbb{R}^2 \to \mathbb{R}^2$ be an abitrary linear map. Can one choose a basis (v_1, v_2) on the domain and a basis (w_1, w_2) on the codomain such that the matrix of T with respect to these bases is diagonal?
  - Yes
  - Rank 0: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
  - Rank 1: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
  - Rank 2: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- Can one choose a basis (v_1, v_2) on both the domain and codomain -- the same basis on both -- such that the matrix of T is diagonal?
  - No
  - $\mathrm{T}(x, y) = (y, 0)$ cannot be diagonal
  - $M(T) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

10/26

Thursday, October 26, 2017

Solving Linear Equations

- Trying to solve the equation
  - $Ax = y$
  - where $x \in V$ is sought, $y \in W$ is given
  - V, W vector spaces
  - $T: V \to W$ linear transformation
- Example 1
  - $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = y_m \end{cases}$
  - Let
    - $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$
    - $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$
    - $A\colon\ \mathbb{R}^n \to \mathbb{R}^n$
    - $A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix}$
    - $A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$
    - A with respect to standard bases of $\mathbb{R}^{\mathrm{n}}, \mathbb{R}^{\mathrm{m}}$
  - Then the linear equations could be represented as
    - $Ax = y$
- Theorem 1
  - Statement
    - If $A: V \to W$ is linear
    - and if $u, v \in V$ are solutions to $Ax = y$
    - (i.e. if $Au = y$, and $Av = y$)
    - Then $u - v \in N(A)$
  - Proof
    - $A(u - v) = Au - Av = y - y = 0$

- Text version
  - If $Ax_p = y$ then for all $x \in V$ with $Ax = y$
  - There is an $x_h \in N(A)$ with $x = x_p + x_h$
- Theorem 2
  - Statement
    - If u is a solution to $Ax = y$
    - and if $w \in N(A)$
    - then u + w is also a solution of $Ax = y$
  - Proof
    - $A(u + w) = Au + Aw = y + 0 = y$
  - Text version
    - For all x_p with $Ax_p = y$ and for all $x_h \in N(A)$
    - $A(x_p + x_h) = y$
- General solution
  - Homogeneous equation

    $Ax = 0$
  - Inhomogeneous equation
    - $Ax = y, \qquad$ where $y \neq 0$
  - The general solution to $Ax = y$ is of the form
    - $x_{gen} = x_p + x_h$, where
    - x_p is a particular solution
    - x_h is the general solution to the homogeneous equation
  - Set of all solutions
    - $\{x \in V | Ax = y\} = \left\{ x_p + x_h \middle| \begin{matrix} Ax_p = y \\ x_h \in N(A) \end{matrix} \right\}$
  - Proof
    - We are given one solution x_p of $Ax = y$
    - If $x_h \in N(A)$
    - then by definition $Ax_h = 0$
    - and hence $A(x_p + x_h) = y$
    - $\Rightarrow x_p + x_h \in \{x \in V | Ax = y\}$
    - Conversely if $Ax = y$ then
    - $A(x - x_p) = Ax - Ax_p = y - y = 0$
    - So $x_h \stackrel{\text{def}}{=} x - x_p \in N(A)$
- Example 2
  - Solve the linear equation $\begin{cases} x_1 + 2x_2 - x_3 = 7 \\ 2x_1 - x_2 + x_3 = 4 \end{cases}$

- Setup
  - $V = \mathbb{R}^3 \Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$
  - $W = \mathbb{R}^2 \Rightarrow y = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$
  - A: $\mathbb{R}^3 \to \mathbb{R}^2$ is matrix multiplication with $\begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix}$
- Range(A)
  - $= \{Ax | x \in \mathbb{R}^3\}$
  - $= \{\text{all possible } y \in \mathbb{R}^2 \text{ for which } Ax = y \text{ has a solution}\}$
- By Rank–nullity theorem
  - $\dim \mathrm{N}(A) + \dim \mathrm{Range}(A) = \dim \mathbb{R}^3 = 3$

dim Range(A)	dim N(A)
0	3
1	2
2	1

- Solving the equation by Gaussian Elimination
  - $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 2 & -1 & 1 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1/5 & 3 \\ 0 & 1 & -3/5 & 2 \end{array}\right]$
  - $\begin{cases} x_1 + \frac{1}{5}x_3 = 3 \\ x_2 - \frac{3}{5}x_3 = 2 \end{cases}$
  - Let $x_3 = 5c$
  - Then $\begin{cases} x_2 = 2 + \frac{3}{5}x_3 = 2 + 3c \\ x_2 = 3 - \frac{1}{5}x_3 = 3 - c \end{cases}$
  - Therefore the general solution is
  - $x = \begin{bmatrix} 3 - c \\ 2 + 3c \\ 5c \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}}_{x_p} + c \underbrace{\begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}}_{x_h}$

- Example 3
  - Given
    - $V = W = \{\text{functions } y: [a, b] \to \mathbb{R}\}$
    - $A: V \to W$ where $Af = f' + xf$
  - Question
    - Solve $\frac{dy}{dx} + xy = x$

- The general solution is in form of
  - $x + x_p + x_h$
- It's easy to find a particular solution
  - $f_p(x) = 1$
- Solving by separating variables
  - $\frac{dy}{dx} + xy = x$
  - $\frac{dy}{dx} = x(1-y)$
  - $\frac{1}{(1-y)} dy = xdx$
  - $\int \frac{1}{(1-y)} dy = \int xdx$
  - $-\ln(-y+1) = \frac{x^2}{2} + c$
  - $f_h(x) = c \cdot e^{-\frac{x^2}{2}}$
- Therefore the general solution is
  - $f(x) = f_h(x) + f_p(x) = c \cdot e^{-\frac{x^2}{2}} + 1$

10/30

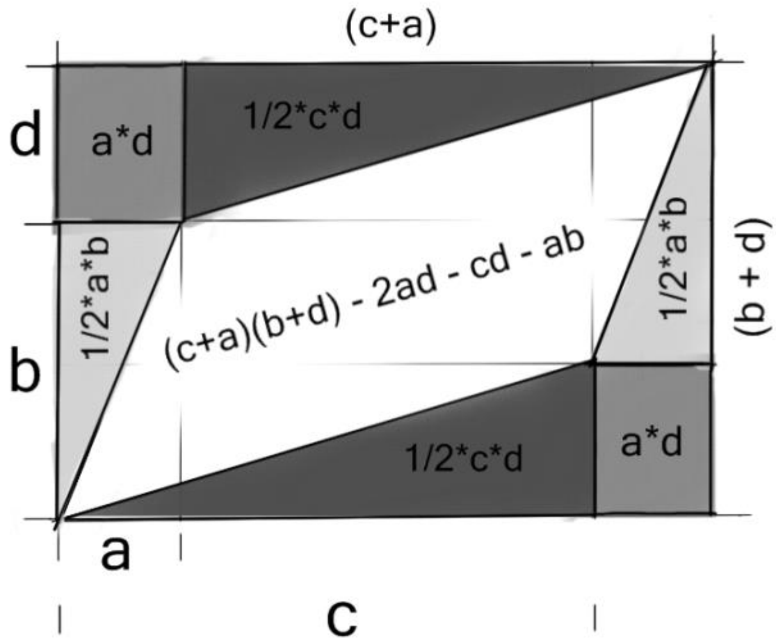
Monday, October 30, 2017

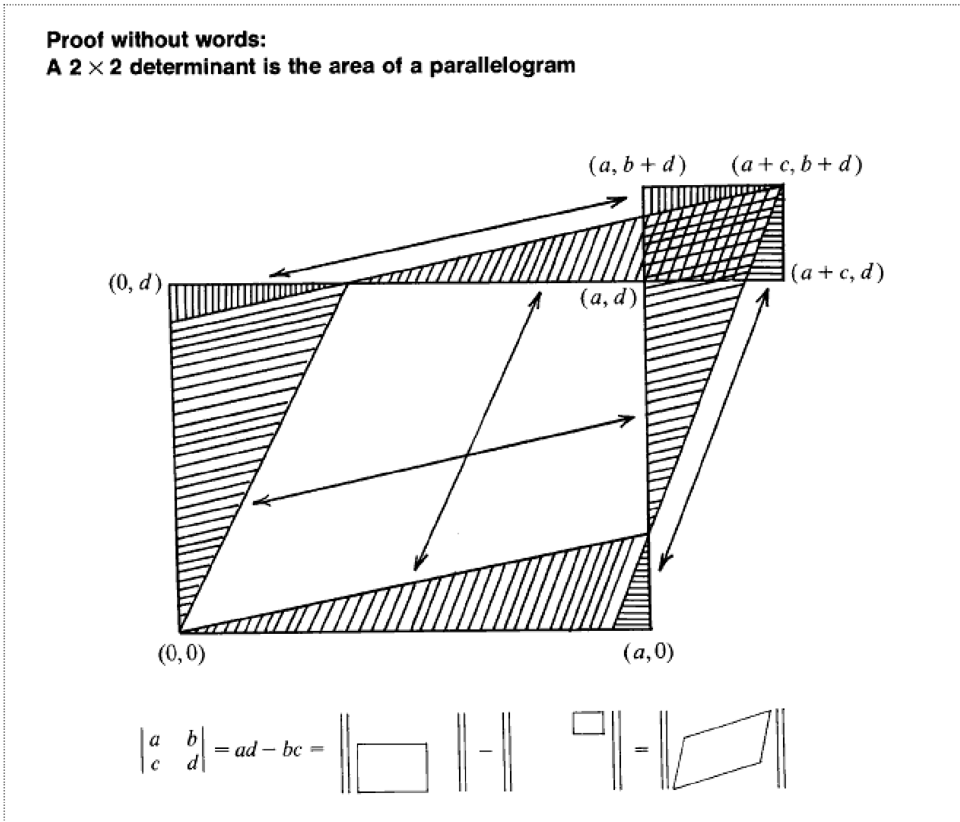
Question 1

- Let A be an $n \times n$ square matrix which has a row or column of all zeros
- Prove: A is singular (i.e. not invertible)
- Proof: Column of all zeros
  - $Ae_i = \begin{pmatrix} * & \dots & 0 & \dots & * \\ \vdots & \dots & \vdots & \dots & \vdots \\ * & \dots & \underbrace{0}_{i\text{-th}} & \dots & * \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \} i\text{-th} = 0$
  - $Ae_i = 0 \Rightarrow A$ is not injective $\Rightarrow A$ is not invertable
- Proof: Rows of all zeros
  - $\forall v \in V \Rightarrow Av = \begin{pmatrix} * & \dots & * \\ 0 & \dots & 0 \\ * & \dots & * \end{pmatrix} v = \begin{pmatrix} \vdots \\ 0 \\ \vdots \end{pmatrix}$
  - $Av = 0 \Rightarrow A$ is not surjective $\Rightarrow A$ is not invertable

Question 2

- Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map.
- Computer the area of the image of the unit square $[0,1]^2$
- i.e. the set $T([0,1]^2) = \{T(x, y): x, y \in [0,1]\} \subseteq \mathbb{R}^2$
- Answer
  - Area of image $= \det(T)$
- Proof





Question 3

- Let V be a finite-dimensional vector space
- Let $T: V \to V$ be a linear map such that $TS = ST$ for all linear maps $S: V \to V$
- Prove that there exists $c \in \mathbb{R}$ such that for all $v \in V$, we have $Tv = cv$
- Prove (Version 1)

- Let $E_{ij} = \begin{bmatrix} 0 & & 0 & & 0 \\ & \ddots & \vdots & \iddots & \\ 0 & \dots & 1 & \dots & 0 \\ & \iddots & \vdots & \ddots & \\ 0 & & \underbrace{0}_{j\text{-th}} & & 0 \end{bmatrix} \}i\text{-th}$, where $i \neq j$
  - $TE_{ij} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n1} \end{bmatrix} \begin{bmatrix} 0 & & 0 & & 0 \\ & \ddots & \vdots & \iddots & \\ 0 & \dots & 1 & \dots & 0 \\ & \iddots & \vdots & \ddots & \\ 0 & & 0 & & 0 \end{bmatrix} = \begin{bmatrix} 0 & \dots & a_{1j} & \dots & 0 \\ \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & a_{jj} & \dots & 0 \\ \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & a_{nj} & \dots & 0 \end{bmatrix}$
  - $E_{ij}T = \begin{bmatrix} 0 & & 0 & & 0 \\ & \ddots & \vdots & \iddots & \\ 0 & \dots & 1 & \dots & 0 \\ & \iddots & \vdots & \ddots & \\ 0 & & 0 & & 0 \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n1} \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{i1} & \dots & a_{ii} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & 0 \end{bmatrix}$
- Because $TS = ST$ for all linear maps $S: V \to V$
  - $TE_{ij} = E_{ij}T$
  - $\begin{bmatrix} 0 & \dots & a_{1j} & \dots & 0 \\ \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & a_{jj} & \dots & 0 \\ \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & a_{nj} & \dots & 0 \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{i1} & \dots & a_{ii} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & 0 \end{bmatrix}$
  - $\Rightarrow \begin{cases} a_{ii} = a_{jj} & \forall i, j \in \{1,2,\dots,n\}, i \neq j \\ a_{kl} = 0 & \forall k, l \in \{1,2,\dots,n\}, k \neq l \end{cases}$
  - Let $a_{11} = a_{22} = \cdots a_{nn} = c$
- Therefore $T = \begin{bmatrix} c & & \\ & \ddots & \\ & & c \end{bmatrix}$ is a scalar matrix i.e. $Tv = cv$
- Also, T satisfied the following property for all linear maps $S: V \to V$
  - $TSv = T(Sv) = c \cdot Sv = S(cv) = STv$

- Proof (Version 2)
  - Assume Tv and v is linearly independent
    - i.e. $Tv \neq cv$
  - Then the following is a basis for V
    - $\{v, Tv, e_1, e_2, \dots\}$
  - Define S to be
    - $S \overset{\text{def}}{=} \begin{cases} S(v) = v \\ S(Tv) = v \\ S(e_1) = 0 \\ S(e_2) = 0 \\ \vdots \end{cases}$
  - Then
    - $T(v) = T(S(v)) = TS(v) = ST(v) = S(Tv) = v$
  - Which makes a contradiction

- Therefore Tv and v is linearly dependent i.e. $Tv = cv$

10/31

Tuesday, October 31, 2017

Example of Determinants

- $\det|a_{11}| = a_{11}$
- $\det\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\,a_{22} - a_{21}a_{12}$
- $\det\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \end{matrix}$

Apostol's Notation for Determinant

- $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$
- $A_i = i\text{-th row of } A = [a_{i1}, a_{i2}, \dots, a_{in}]$
- $A^i = i\text{-th column of } A = [a_{1i}, a_{2i}, \dots, a_{ni}]^T$
- $I_1 = \text{1st row of } I = [1,0,\dots,0]$
- $I^1 = \text{1st column of } I = [1,0,\dots,0]^T$

Properties of Determinant

- $\det A = d(A_1, A_2, \dots, A_n)$
- Linearity
  - $d(B + C, A_2, \dots, A_n) = d(B, A_2, \dots, A_n) + d(C, A_2, \dots, A_n)$
  - $d(tA_1, A_2, \dots, A_n) = d(A_1, A_2, \dots, A_n)$
- Alternating
  - $d(A_1, A_2, \dots, A_i, .., A_j, \dots A_n) = -d(A_1, A_2, \dots, A_j, .., A_i, \dots A_n)$
- Identity
  - $\det(I) = \det\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = 1$
- Note
  - $d(A, B + C)$
  - $= -\,d(B + C, A)$
  - $= -\,d(B, A) - d(C, A)$
  - $= d(A, B) + d(A, C)$
- Fact
  - $d(A_1, A_2, \dots, A_n) = 0 \; if \; A_i = A_j \text{ for some } i \neq j$

- Proof by Alternating Properties

Example using Properties

- $\det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = d(A_1, A_2)$
- $= d(a_1 I_1 + a_2 I_2, b_1 I_1 + b_2 I_2)$
- $= a_1 b_1 \cdot d(I_1, I_1) + a_1 b_2 \cdot d(I_1, I_2) + a_2 b_1 \cdot d(I_2, I_1) + a_2 b_2 \cdot d(I_2, I_2)$
- $= \underbrace{a_1 b_1 \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}_{=0} + a_1 b_2 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + a_2 b_1 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \underbrace{a_2 b_2 \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}}_{=0}$
- $= a_1 b_2 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + a_2 b_1 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$
- $= a_1 b_2 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - a_2 b_1 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$
- $= (a_1 b_2 - a_2 b_1) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$
- $= a_1 b_2 - a_2 b_1$

Formula

- Expanding $\det A$
  - $\det A = d(A_1, \ldots A_n)$
  - $= d\left(\sum_{i_1=1}^{n} a_{1i_1} I_{i_1}, \sum_{i_2=1}^{n} a_{2i_2} I_{i_2}, \ldots, \sum_{i_n=1}^{n} a_{1i_n} I_{i_n}\right)$
  - $= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \ldots \sum_{i_n=1}^{n} a_{1i_1} a_{2i_2} \ldots a_{ni_n} \cdot d\left(I_{i_1}, I_{i_2}, \ldots, I_{i_n}\right)$
- Consider the term with indices $i_1, i_2, \ldots, i_n$
  - If any two of these numbers are equal, then
  - $d\left(I_{i_1}, I_{i_2}, \ldots, I_{i_n}\right) = 0$
- Reduce 0 terms
  - $\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \ldots \sum_{i_n=1}^{n} a_{1i_1} a_{2i_2} \ldots a_{ni_n} \cdot d\left(I_{i_1}, I_{i_2}, \ldots, I_{i_n}\right)$
  - $= \sum_{\substack{1 \le i_1, i_2, \ldots, i_n \le n \\ \text{all different}}}^{n} a_{1i_1} a_{2i_2} \ldots a_{ni_n} \cdot d\left(I_{i_1}, I_{i_2}, \ldots, I_{i_n}\right)$
  - $= \sum_{\substack{Let\ (i_1, i_2, \ldots, i_n) \text{ to be the} \\ \text{permutation of } (1,2,\ldots n)}}^{n} a_{1i_1} a_{2i_2} \ldots a_{ni_n} \cdot d\left(I_{i_1}, I_{i_2}, \ldots, I_{i_n}\right)$
- You can sort a permutation using simple exchange
  - i.e. $d(I_1, I_4, I_3, I_5, I_2) = -d(I_1, I_2, I_3, I_5, I_4) = d(I_1, I_2, I_3, I_4, I_5) = 1$

- Using this property, we get

$$d\left(I_{i_1}, I_{i_2}, \ldots, I_{i_n}\right) = \begin{cases} 1 & \text{even arrangement of } (i_1, i_2, \ldots, i_n) \\ -1 & \text{odd arrangement of } (i_1, i_2, \ldots, i_n) \end{cases}$$

Examples

- $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 0 & 1 & 4 \end{vmatrix} = 0$, because of two euqal rows
- $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 7 \end{vmatrix} = 1 \times 3 \times 4 \times 7 = 84$
- Given $\det(A_{3\times3}) = 5 \Rightarrow \det(2A) = 2^3 \times 5 = 40$