Math 632 004 HW03

Shawn Zhong

TOTAL POINTS

88 / 90

QUESTION 1

1 Problem 1: Expected number of visits **15 / 15**

✓ **- 0 pts** Correct

- 2 pts Minor mistake in computing $$\rho_{2,3}$$

- 2 pts Minor mistake in computing $$\rho_{3,3}$$

- 1 pts Minor mistake in formula for $$E_2[N(3)]$$

- 3 pts Mistake in formula for $$E_3[N(3)]$$

- 4 pts Claim that 4 is transient

- 4 pts No work submitted for (c)

QUESTION 2

2 Problem 2: Success run chain **25 / 25**

- 2 pts (a) Mistake in series summation to show 0 is recurrent

- 1 pts (a) Forgot to justify recurrence for all other states

- 2 pts (b) Incorrect recurrence for stationary measure

- 2 pts (b) Series summation is incorrect

- 1 pts (b) Wrong name for the distribution

- 2 pts (c) Incorrect recurrence for stationary measure

- 2 pts (c) Series summation is incorrect for stationary measure

- 1 pts (c) Mistake in computation of $$E_0[T_0]$$

- 4 pts (d) Did not acknowledge the stationary distribution in any way

- 2 pts (d) Mistake in how stationary distribution is applied

- 4 pts (e) Came to the wrong conclusion

✓ **- 0 pts** Correct

- 25 pts No work submitted

QUESTION 3

3 Problem 3: Reflected random walk **15 / 15**

- 4 pts (a) Serious error in recursion of the invariant measure

- 2 pts (a) Minor error in recursion of the invariant measure

- 3 pts (a) Series summation has serious error

- 2 pts (b) Wrong conclusion for $$\alpha < 1/2$$

- 2 pts (b) Wrong conclusion for $$\alpha > 1/2$$

- 1 pts (b) Wrong conclusion for $$\alpha = 1/2$$

✓ **- 0 pts All parts correct**

QUESTION 4

4 Problem 4: Symmetric simple random walk **8 / 10**

- 0 pts Correct

✓ **- 2 pts Correct that the only stationary measures are constant, but you must prove these are the only ones.**

- 4 pts Did not get the correct answer.

QUESTION 5

5 Problem 5: Durrett 1.13 **15 / 15**

✓ **- 0 pts** Correct

- 3 pts (a) Some mistakes in finding $$p^2$$

- 2 pts (b) Some mistake in the stationary distribution of $$p$$

- 2 pts (b) Some mistake in the stationary distribution of $$p^2$$

- 1 pts (c) Missed one of the limits or had a wrong answer with one of the limits

- 2 pts (c) No work submitted or wrong answers with both limits

QUESTION 6

6 Problem 6: Durrett 1.32 **10 / 10**

✓ **- 0 pts** Correct

- 4 pts Did not recognize the limit is the stationary distribution

- 4 pts Did not find correct stationary distribution

HW3 - Problem & Answer

Friday, September 28, 2018 4:47 PM

Math 632 Lecture 4, Fall 2018, Homework 3

Due Friday October 5 by 2 PM.

There are 6 problems altogether in this homework set. 4 are stated below and 2 problems from Durrett's textbook. Please check the homework instructions on the course homepage. In particular:

- Credit comes from your reasoning, not your numerical answer.
- Observe rules of academic integrity. You are encouraged to discuss the problems with fellow students, but copied work is not acceptable and will result in zero credit.
- You may find solutions to some exercises on the web. However, the point of the homework is to give you the problem solving practice you need in the exams. Hence it is not smart to take shortcuts to secure the few points that come from homework.

Question 1

1. Consider the discrete time Markov chain with the state space $S = \{1,2,3,4,5\}$ and the transition matrix:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0.2 & 0 & 0.4 & 0.2 & 0.2 \\ 0.3 & 0.1 & 0 & 0.2 & 0.4 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{bmatrix}.$$

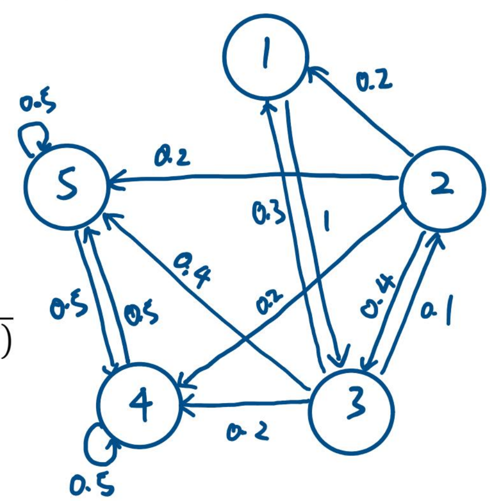
Let $N(y)$ be the total number of visits to the state y for all time, excluding a possible visit at time zero. In symbols,

$$N(y) = \sum_{n=1}^{\infty} \mathbf{1}_{\{X_n = y\}}.$$

(a) Calculate $E_2[N(3)]$.

(b) Calculate $E_3[N(3)]$.

(c) Calculate $E_4[N(4)]$.



- $\mathbb{E}_2[N(3)] = \dfrac{\rho_{23}}{1-\rho_{33}}$

$$= \frac{1 - p(2,5) - p(2,4)}{p(3,4) + p(3,5) + p(3,2)\big(p(2,5) + p(2,4)\big)}$$

$$= \frac{1 - 0.2 - 0.2}{0.2 + 0.4 + 0.1 \times (0.2 + 0.2)} = 0.9375$$

- $\mathbb{E}_3[N(3)] = \dfrac{\rho_{33}}{1-\rho_{33}} = \dfrac{1}{1-\rho_{33}} - 1$

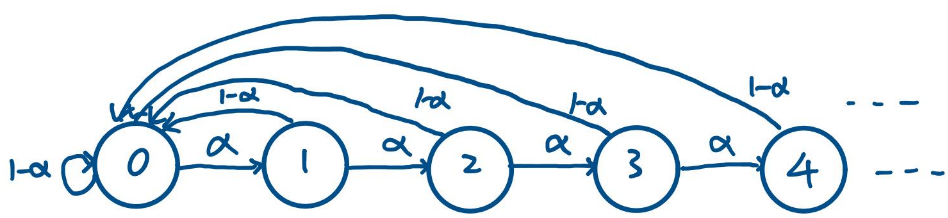
$$= \frac{1}{0.2 + 0.4 + 0.1 \times (0.2 + 0.2)} - 1 = 0.5625$$

- By the graph above, state 4 is recurrent, so $\mathbb{E}_4[N(4)] = +\infty$

Question 2

2. Success run chain. Imagine a never ending succession of free throws with a basketball. Each throw is independently either a basket with probability α or a miss with probability $1-\alpha$. Assume $0 < \alpha < 1$. After the nth throw, let X_n be the number of consecutive baskets since the last miss, in other words the length of the current success run. X_n is called the *success run chain*. For example, if throw 7 is a miss, throws 8, 9, and 10 are baskets, and throw 11 again a miss, then $X_7 = 0$, $X_8 = 1$, $X_9 = 2$, $X_{10} = 3$, and $X_{11} = 0$. Thus every basket increases X_n by 1, and every miss sends X_n back to 0. X_n is a Markov chain with state space $\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \dots\}$.

(a) Draw the transition graph of the success run chain and give its transition probability. Classify states according to recurrence/transience. (Note that the state space is infinite so recurrence/transience is not a trivial matter.)



- $\mathbb{P}_0(T_0 = \infty) = \prod_{n=0}^{\infty} p(n, n+1) = \prod_{n=0}^{\infty} \alpha = 0$, so 0 is recurrent
- For $n \geq 1$, $p^n(0, n) = \alpha^n > 0$ (*i.e.* $0 \Rightarrow n$), so n is also recurrnet
- Therefore all states is recurrent

(b) Find the invariant distribution or determine that one does not exist. If you propose an invariant distribution, check that the probabilities add up to 1. Is it a familiar probability distribution with a name?

- Let μ be an invariant measure
- $\mu(k) = \sum_{l=0}^{\infty} \mu(l) p(l, k) = \mu(k-1) p(k-1, k) = \mu(k-1)\alpha = \mu(0)\alpha^k$, for $k \geq 1$
- Define $\pi(k) \coloneqq \frac{\mu(k)}{\sum_{k=0}^{\infty} \mu(k)} = \frac{\mu(0)\alpha^k}{\mu(0)\sum_{k=0}^{\infty} \alpha^k} = (1-\alpha)\alpha^k$
- $\sum_{k=0}^{\infty} \pi(k) = (1-\alpha) \sum_{k=0}^{\infty} \alpha^k = 1$, so $\pi(k)$ is an invariant distribution
- π is a geometric distribution with parameter α

1 Problem 1: Expected number of visits **15 / 15**

- ✓ **- 0 pts** Correct
- **- 2 pts** Minor mistake in computing $$\rho_{2,3}$$
- **- 2 pts** Minor mistake in computing $$\rho_{3,3}$$
- **- 1 pts** Minor mistake in formula for $$E_2[N(3)]$$
- **- 3 pts** Mistake in formula for $$E_3[N(3)]$$
- **- 4 pts** Claim that 4 is transient
- **- 4 pts** No work submitted for (c)

$$= \frac{1}{0.2 + 0.4 + 0.1 \times (0.2 + 0.2)} - 1 = 0.5625$$

- By the graph above, state 4 is recurrent, so $\mathbb{E}_4[N(4)] = +\infty$

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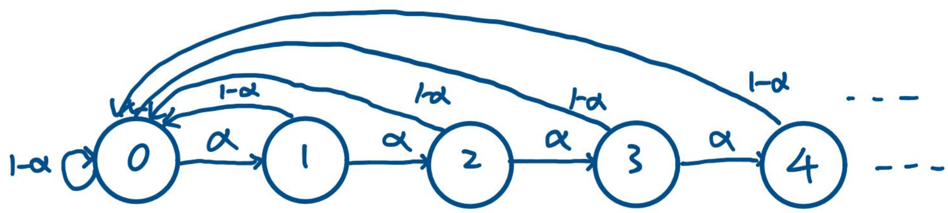
2. Success run chain. Imagine a never ending succession of free throws with a basketball. Each throw is independently either a basket with probability α or a miss with probability $1-\alpha$. Assume $0 < \alpha < 1$. After the nth throw, let X_n be the number of consecutive baskets since the last miss, in other words the length of the current success run. X_n is called the *success run chain.* For example, if throw 7 is a miss, throws 8, 9, and 10 are baskets, and throw 11 again a miss, then $X_7 = 0$, $X_8 = 1$, $X_9 = 2$, $X_{10} = 3$, and $X_{11} = 0$. Thus every basket increases X_n by 1, and every miss sends X_n back to 0. X_n is a Markov chain with state space $\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \dots\}$.

(a) Draw the transition graph of the success run chain and give its transition probability. Classify states according to recurrence/transience. (Note that the state space is infinite so recurrence/transience is not a trivial matter.)



- $\mathbb{P}_0(T_0 = \infty) = \prod_{n=0}^{\infty} p(n, n+1) = \prod_{n=0}^{\infty} \alpha = 0$, so 0 is recurrent
- For $n \geq 1$, $p^n(0, n) = \alpha^n > 0$ (*i.e.* $0 \Rightarrow n$), so n is also recurrnet
- Therefore all states is recurrent

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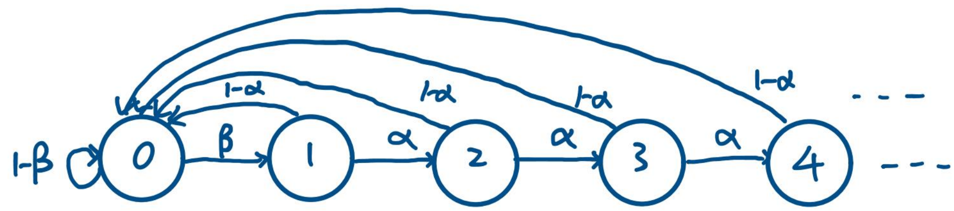
- Let μ be an invariant measure
- $\mu(k) = \sum_{l=0}^{\infty} \mu(l) p(l, k) = \mu(k-1) p(k-1, k) = \mu(k-1)\alpha = \mu(0)\alpha^k$, for $k \geq 1$
- Define $\pi(k) := \frac{\mu(k)}{\sum_{k=0}^{\infty} \mu(k)} = \frac{\mu(0)\alpha^k}{\mu(0)\sum_{k=0}^{\infty} \alpha^k} = (1-\alpha)\alpha^k$
- $\sum_{k=0}^{\infty} \pi(k) = (1-\alpha) \sum_{k=0}^{\infty} \alpha^k = 1$, so $\pi(k)$ is an invariant distribution
- π is a geometric distribution with parameter α

(c) Change the model slightly. Let $0 < \beta < 1$ be another parameter and suppose that immediately after a miss, the next throw succeeds with probability β. But as before, after a successful free throw, the next shot succeeds with probability α. Find again the invariant distribution π. Find also all the probabilities $P_0(T_0 = k)$ for positive integers k. Using these probabilities compute the expectation $E_0[T_0]$. Check that your answers satisfy the identity $1/\pi(0) = E_0[T_0]$ as the theory states.

Hint. You may need to evaluate a series of the type $\sum_{k=1}^{\infty} kx^{k-1}$ for $|x| < 1$. There are several ways to do this. One goes by $\sum_{k=1}^{\infty} kx^{k-1} = \sum_{k=1}^{\infty} \sum_{j=1}^{k} x^{k-1}$ and switches the order of summation. Another observes that $\sum_{k=1}^{\infty} kx^{k-1} = \sum_{k=0}^{\infty} \frac{d}{dx} x^k = \frac{d}{dx} \left(\sum_{k=0}^{\infty} x^k\right)$, evaluates the series and then differentiates.



- Let μ be an invariant measure
  - $\mu(k) = \sum_{l=0}^{\infty} \mu(l)p(l,k) = \mu(k-1)p(k-1,k)$
  - For $k = 1, \mu(1) = \mu(0)\beta$
  - For $k \geq 2, \mu(k) = \mu(k-1)\alpha = \mu(0)\alpha^{k-1}\beta$
  - Therefore $\mu(k) = \mu(0)\alpha^{k-1}\beta$, for $k \geq 1$
- Normalize μ
  - $\sum_{k=0}^{\infty} \mu(k) = \mu(0) + \mu(0)\beta \sum_{k=1}^{\infty} \alpha^{k-1} = \mu(0)\left(1 + \frac{\beta}{1-\alpha}\right) = \mu(0)\left(\frac{1-\alpha+\beta}{1-\alpha}\right)$
  - For $k \geq 1, \pi(k) = \frac{\mu(k)}{\sum_{k=0}^{\infty} \mu(k)} = \frac{\mu(0)\alpha^{k-1}\beta}{\mu(0)\left(\frac{1-\alpha+\beta}{1-\alpha}\right)} = \frac{\alpha^{k-1}\beta(1-\alpha)}{1-\alpha+\beta}$
  - For $k = 0, \pi(0) = \frac{\mu(0)}{\sum_{k=0}^{\infty} \mu(k)} = \frac{\mu(0)}{\mu(0)\left(\frac{1-\alpha+\beta}{1-\alpha}\right)} = \frac{1-\alpha}{1-\alpha+\beta}$
- $\mathbb{E}_0[T_0] = \sum_{k=1}^{\infty} k\mathbb{P}_0(T_0 = k)$

$$= \mathbb{P}_0(T_0 = 1) + \sum_{k=2}^{\infty} k\mathbb{P}_0(T_0 = k)$$

$$= 1 - \beta + \sum_{k=2}^{\infty} k\beta\alpha^{k-2}(1-\alpha)$$

$$= 1 - \beta + (1-\alpha)\beta \sum_{k=2}^{\infty} k\alpha^{k-2}$$

$$= 1 - \beta + (1-\alpha)\beta\left(\frac{d}{d\alpha}\sum_{k=2}^{\infty}\alpha^{k-2} + \sum_{k=2}^{\infty}\alpha^{k-2}\right)$$

$$= 1 - \beta + (1-\alpha)\beta\left(\frac{d}{d\alpha}\left(\frac{1}{1-\alpha}\right) + \frac{1}{1-\alpha}\right)$$

$$= 1 - \beta + (1-\alpha)\beta\left(\frac{1}{(1-\alpha)^2} + \frac{1}{1-\alpha}\right)$$

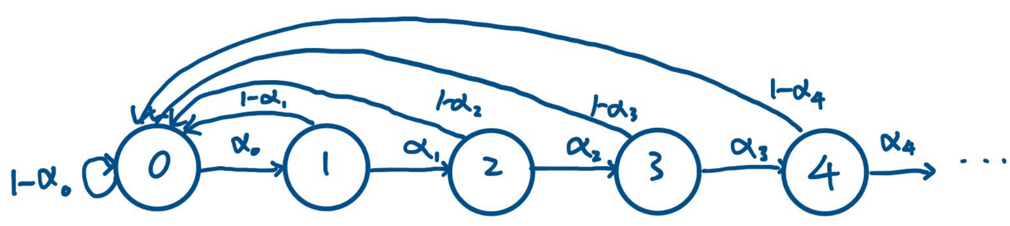
$$= 1 - \beta + \beta\frac{2-\alpha}{(1-\alpha)} = \frac{1-\alpha+\beta}{1-\alpha} = \frac{1}{\pi(0)}$$

(d) Imagine that the chain of part (c) has been running for a very long time. What is the probability that the next two shots succeed?

Hint. For a sanity check of your answer, see that if $\beta = \alpha$ your answer is consistent with the initial description of the free throws.

- $\sum_{k=0}^{\infty}\pi(k)p(k,k+1)p(k+1,k+2)$
- $= \pi(0)p(0,1)p(1,2) + \sum_{k=1}^{\infty}\pi(k)p(k,k+1)p(k+1,k+2)$
- $= \frac{1-\alpha}{1-\alpha+\beta}\beta\alpha + \sum_{k=1}^{\infty}\frac{\alpha^{k-1}\beta(1-\alpha)}{1-\alpha+\beta}\alpha^2 = \frac{1-\alpha}{1-\alpha+\beta}\beta\alpha + \frac{\alpha^2\beta(1-\alpha)}{1-\alpha+\beta}\sum_{k=0}^{\infty}\alpha^k$
- $= \frac{1-\alpha}{1-\alpha+\beta}\beta\alpha + \frac{\alpha^2\beta(1-\alpha)}{1-\alpha+\beta}\cdot\frac{1}{1-\alpha} = \frac{\alpha\beta-\alpha^2\beta+\alpha^2\beta}{1-\alpha+\beta} = \frac{\alpha\beta}{1-\alpha+\beta}$

(e) Now suppose that we have parameters $\alpha_0, \alpha_1, \alpha_2, \ldots$, all strictly between 0 and 1, and we specify that, after k consecutive successes, the next shot succeeds with probability α_k. That is, now the transition probabilities are $p(k,k+1) = \alpha_k$, $p(k,0) = 1-\alpha_k$. Is it possible to choose the numbers $\alpha_k \in (0,1)$ so that the MC is transient?



- Let $\alpha_i = \exp\left(-\frac{1}{2^i}\right)$, then $\prod_{i=0}^{\infty}\alpha_i = \prod_{i=0}^{\infty}\exp\left(-\frac{1}{2^i}\right) = \exp\left(-\sum_{i=0}^{\infty}\frac{1}{2^i}\right) = \exp(-2)$
- For $n = 0$, $\mathbb{P}_0(T_0 = \infty) \geq \prod_{i=0}^{\infty} p(i,i+1) = \prod_{i=0}^{\infty}\alpha_i = \exp(-2) > 0$, so state 0 is transient
- For $n \geq 1$, $\begin{cases}\mathbb{P}_n(T_0 = \infty) = \prod_{i=n}^{\infty}\alpha_i > 0 \\ p^n(0,n) = \prod_{i=0}^{n}\alpha_i > 0\end{cases} \Rightarrow \begin{cases}\rho_{n0} < 1 \\ 0 \Rightarrow n\end{cases} \Rightarrow$ state n is transient

2 Problem 2: Success run chain **25 / 25**

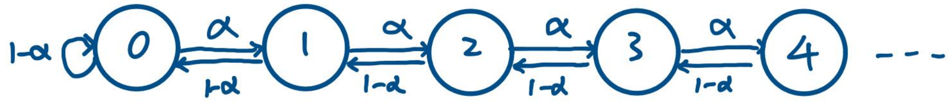
- **- 2 pts** (a) Mistake in series summation to show 0 is recurrent
- **- 1 pts** (a) Forgot to justify recurrence for all other states
- **- 2 pts** (b) Incorrect recurrence for stationary measure
- **- 2 pts** (b) Series summation is incorrect
- **- 1 pts** (b) Wrong name for the distribution
- **- 2 pts** (c) Incorrect recurrence for stationary measure
- **- 2 pts** (c) Series summation is incorrect for stationary measure
- **- 1 pts** (c) Mistake in computation of $$E_0[T_0]$$
- **- 4 pts** (d) Did not acknowledge the stationary distribution in any way
- **- 2 pts** (d) Mistake in how stationary distribution is applied
- **- 4 pts** (e) Came to the wrong conclusion
- ✓ **- 0 pts** Correct
- **- 25 pts** No work submitted

Question 3

3. Reflected random walk. Fix a parameter $0 < \alpha < 1$. Consider the Markov chain with state space $\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \dots\}$ and transition probability

$$p(0,0) = 1 - \alpha, \quad p(x, x+1) = \alpha \text{ for } x \geq 0, \quad \text{and } p(x, x-1) = 1 - \alpha \text{ for } x \geq 1.$$

(a) Specify for which values of α there exists an invariant distribution. Find the invariant distribution for those values of α for which it exists.



- Let μ be an invariant measure
  - For $k = 0, \mu(0) = \sum_{l=0}^{\infty} \mu(l)p(l,0) = \mu(0)p(0,0) + \mu(1)p(1,0) = (1-\alpha)\big(\mu(0) + \mu(1)\big)$
  - For $k \geq 1, \mu(k) = \sum_{l=0}^{\infty} \mu(l)p(l,k)$

    $= \mu(k-1)p(k-1,k) + \mu(k+1)p(k+1,k)$

    $= \alpha \cdot \mu(k-1) + (1-\alpha) \cdot \mu(k+1)$
  - Set $\mu(0) = 1$, then $\mu(1) = \frac{\alpha}{1-\alpha}, \mu(2) = \left(\frac{\alpha}{1-\alpha}\right)^2, \cdots$
  - Therefore, $\mu(k) = \left(\frac{\alpha}{1-\alpha}\right)^k$
- Normalized μ
  - $\sum_{k=0}^{\infty} \mu(k) = \sum_{k=0}^{\infty} \left(\frac{\alpha}{1-\alpha}\right)^k = \frac{1}{1 - \frac{\alpha}{1-\alpha}} = \frac{\alpha - 1}{2\alpha - 1}$, for $0 < \left|\frac{\alpha}{1-\alpha}\right| < 1 \Rightarrow 0 < \alpha < 0.5$
  - $\pi(k) = \frac{\mu(k)}{\sum_{k=0}^{\infty} \mu(k)} = \frac{2\alpha - 1}{\alpha - 1}\left(\frac{\alpha}{1-\alpha}\right)^k$ is a invariant distribution given $\alpha \in (0, 0.5)$

(b) For which values of α is the MC recurrent and for which values transient? Here you probably cannot give a rigorous proof, but you should be able to give an intuitively plausible argument. Note that when you are away from the origin, the reflected random walk behaves just like ordinary random walk.

- For $0 < \alpha \leq 0.5$, the MC is recurrent
  - The chain is moving towards left, so state 0 is recurrent
  - Furthermore, $0 \Rightarrow n, \forall n \geq 1$, so the entire chain is recurrent
- For $0.5 < \alpha < 1$, the MC is transient
  - All states are moving rightward to infity, so it's impossible to comeback
  - Therefore the chain is transient

3 Problem 3: Reflected random walk **15 / 15**

- **- 4 pts** (a) Serious error in recursion of the invariant measure
- **- 2 pts** (a) Minor error in recursion of the invariant measure
- **- 3 pts** (a) Series summation has serious error
- **- 2 pts** (b) Wrong conclusion for $$\alpha < 1/2$$
- **- 2 pts** (b) Wrong conclusion for $$\alpha > 1/2$$
- **- 1 pts** (b) Wrong conclusion for $$\alpha = 1/2$$
- ✓ **- 0 pts** **All parts correct**

Question 4

4. Symmetric simple random walk. Consider the Markov chain with state space $S = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. The transition probabilities for this chain are

$$p(j, j-1) = \frac{1}{2}, \quad p(j, j+1) = \frac{1}{2}$$

for all $j \in S$. Find all the stationary measures and stationary distributions for this Markov chain.

- Let μ be an invariant measure
- $\mu(k) = \sum_{l=-\infty}^{\infty} \mu(l)p(l,k) = \mu(k-1)p(k-1,k) + \mu(k+1)p(k+1,k) = \frac{\mu(k-1)+\mu(k+1)}{2}$
- Solving the recurrence, we have $\mu(k) = c$, where $c \in \mathbb{R}$ are constants
- Stationary distribution does not exist, since $\sum_{k=1}^{\infty} \mu(k) = +\infty$

Question 5 (Durrett 1.13)

1.13. Consider the Markov chain with transition matrix:

	1	**2**	**3**	**4**
1	0	0	0.1	0.9
2	0	0	0.6	0.4
3	0.8	0.2	0	0
4	0.4	0.6	0	0

(a) Compute p^2. (b) Find the stationary distributions of p and all of the stationary distributions of p^2. (c) Find the limit of $p^{2n}(x,x)$ as $n \to \infty$.

- $\mathcal{P}^2 = \begin{bmatrix} 0.44 & 0.56 & 0 & 0 \\ 0.64 & 0.36 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix}$
- Let $\pi\mathcal{P} = \pi$
  - $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0.6 & 0.4 \\ 0.8 & 0.2 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}^T \Rightarrow \begin{cases} 0.8x_3 + 0.4x_4 = x_1 \\ 0.2x_3 + 0.6x_4 = x_2 \\ 0.1x_1 + 0.6x_2 = x_3 \\ 0.9x_1 + 0.4x_2 = x_4 \end{cases} \Rightarrow \pi = \begin{bmatrix} 4/15 \\ 7/30 \\ 1/6 \\ 1/3 \end{bmatrix}^T$
- Let $\pi\mathcal{P}^2 = \pi$
  - $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}^T \begin{bmatrix} 0.44 & 0.56 & 0 & 0 \\ 0.64 & 0.36 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}^T \Rightarrow \begin{cases} 0.44x_1 + 0.64x_2 = x_1 \\ 0.56x_1 + 0.36x_2 = x_2 \\ 0.2x_3 + 0.4x_4 = x_3 \\ 0.8x_3 + 0.6x_4 = x_4 \end{cases} \Rightarrow \begin{cases} x_2 = \frac{7}{8}x_1 \\ x_4 = 2x_3 \end{cases}$
  - $\Rightarrow \pi = \begin{bmatrix} \frac{8(1-c)}{15} & \frac{7(1-c)}{15} & \frac{1}{3}c & \frac{2}{3}c \end{bmatrix}$, for $c \in [0,1]$
- For $\mathcal{P}^2$, $\{1,2\}$ is an irreducible closed set
  - $\pi_{\{1,2\}} = \begin{bmatrix} \frac{8}{15} & \frac{7}{15} \end{bmatrix} \Rightarrow \lim_{n\to\infty} p^{2n}(1,1) = \frac{8}{15}, \lim_{n\to\infty} p^{2n}(2,2) = \frac{7}{15}$
- For $\mathcal{P}^2$, $\{3, 4\}$ is an irreducible closed set

4 Problem 4: Symmetric simple random walk **8 / 10**

- **- 0 pts** Correct
- ✓ **- 2 pts Correct that the only stationary measures are constant, but you must prove these are the only ones.**
- **- 4 pts** Did not get the correct answer.

Question 4

4. Symmetric simple random walk. Consider the Markov chain with state space $S = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. The transition probabilities for this chain are

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for all $j \in S$. Find all the stationary measures and stationary distributions for this Markov chain.

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4	0.4	0.6	0	0

(a) Compute p^2. (b) Find the stationary distributions of p and all of the stationary distributions of p^2. (c) Find the limit of $p^{2n}(x,x)$ as $n \to \infty$.

- $\mathcal{P}^2 = \begin{bmatrix} 0.44 & 0.56 & 0 & 0 \\ 0.64 & 0.36 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix}$
- Let $\pi\mathcal{P} = \pi$
  - $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0.6 & 0.4 \\ 0.8 & 0.2 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}^T \Rightarrow \begin{cases} 0.8x_3 + 0.4x_4 = x_1 \\ 0.2x_3 + 0.6x_4 = x_2 \\ 0.1x_1 + 0.6x_2 = x_3 \\ 0.9x_1 + 0.4x_2 = x_4 \end{cases} \Rightarrow \pi = \begin{bmatrix} 4/15 \\ 7/30 \\ 1/6 \\ 1/3 \end{bmatrix}^T$
- Let $\pi\mathcal{P}^2 = \pi$
  - $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}^T \begin{bmatrix} 0.44 & 0.56 & 0 & 0 \\ 0.64 & 0.36 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}^T \Rightarrow \begin{cases} 0.44x_1 + 0.64x_2 = x_1 \\ 0.56x_1 + 0.36x_2 = x_2 \\ 0.2x_3 + 0.4x_4 = x_3 \\ 0.8x_3 + 0.6x_4 = x_4 \end{cases} \Rightarrow \begin{cases} x_2 = \frac{7}{8}x_1 \\ x_4 = 2x_3 \end{cases}$
  - $\Rightarrow \pi = \begin{bmatrix} \frac{8(1-c)}{15} & \frac{7(1-c)}{15} & \frac{1}{3}c & \frac{2}{3}c \end{bmatrix}$, for $c \in [0,1]$
- For $\mathcal{P}^2$, $\{1,2\}$ is an irreducible closed set
  - $\pi_{\{1,2\}} = \begin{bmatrix} \frac{8}{15} & \frac{7}{15} \end{bmatrix} \Rightarrow \lim_{n\to\infty} p^{2n}(1,1) = \frac{8}{15}, \lim_{n\to\infty} p^{2n}(2,2) = \frac{7}{15}$
- For $\mathcal{P}^2$, $\{3, 4\}$ is an irreducible closed set

- $\pi_{\{3,4\}} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix} \Rightarrow \lim_{n\to\infty} p^{2n}(3,3) = \frac{1}{3}, \lim_{n\to\infty} p^{2n}(4,4) = \frac{2}{3}$

Question 6 (Durrett 1.32)

1.32. The weather in a certain town is classified as rainy, cloudy, or sunny and changes according to the following transition probability is

	R	**C**	**S**
R	1/2	1/4	1/4
C	1/4	1/2	1/4
S	1/2	1/2	0

In the long run what proportion of days in this town are rainy? cloudy? sunny?

- $$\begin{cases} \pi\mathcal{P} = \pi \\ \sum\pi = 1 \end{cases} \Rightarrow \begin{cases} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.5 & 0.5 & 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \\ x_1 + x_2 + x_3 = 1 \end{cases} \Rightarrow \begin{cases} 0.5x_1 + 0.25x_2 + 0.5x_3 = x_1 \\ 0.25x_1 + 0.5x_2 + 0.5x_3 = x_2 \\ 0.25x_1 + 0.25x_2 = x_3 \\ x_1 + x_2 + x_3 = 1 \end{cases} \Rightarrow \pi = \begin{bmatrix} 0.4 \\ 0.4 \\ 0.2 \end{bmatrix}^T$$
- So 40% of days are rainy, 40% of days are cloudy, 20% of days are sunny

5 Problem 5: Durrett 1.13 **15 / 15**

✓ **- 0 pts** Correct

- 3 pts (a) Some mistakes in finding $$p^2$$

- 2 pts (b) Some mistake in the stationary distribution of $$p$$

- 2 pts (b) Some mistake in the stationary distribution of $$p^2$$

- 1 pts (c) Missed one of the limits or had a wrong answer with one of the limits

- 2 pts (c) No work submitted or wrong answers with both limits

- $\pi_{\{3,4\}} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix} \Rightarrow \lim_{n\to\infty} p^{2n}(3,3) = \frac{1}{3}, \lim_{n\to\infty} p^{2n}(4,4) = \frac{2}{3}$

Question 6 (Durrett 1.32)

1.32. The weather in a certain town is classified as rainy, cloudy, or sunny and changes according to the following transition probability is

	R	**C**	**S**
R	1/2	1/4	1/4
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In the long run what proportion of days in this town are rainy? cloudy? sunny?

- $\begin{cases} \pi\mathcal{P} = \pi \\ \sum\pi = 1 \end{cases} \Rightarrow \begin{cases} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.5 & 0.5 & 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \\ x_1 + x_2 + x_3 = 1 \end{cases} \Rightarrow \begin{cases} 0.5x_1 + 0.25x_2 + 0.5x_3 = x_1 \\ 0.25x_1 + 0.5x_2 + 0.5x_3 = x_2 \\ 0.25x_1 + 0.25x_2 = x_3 \\ x_1 + x_2 + x_3 = 1 \end{cases} \Rightarrow \pi = \begin{bmatrix} 0.4 \\ 0.4 \\ 0.2 \end{bmatrix}^T$
- So 40% of days are rainy, 40% of days are cloudy, 20% of days are sunny

6 Problem 6: Durrett 1.32 **10 / 10**

✓ **- 0 pts** **Correct**

- 4 pts Did not recognize the limit is the stationary distribution

- 4 pts Did not find correct stationary distribution